Computational Modeling of the Cardiovascular System

Finite Element Method I

Mechanical Modeling of Tissues II

CVRTI

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Overview

- Finite Element Method I
  - Shape Functions
  - Element Matrix and Vector
  - Gaussian Quadrature

Group work & pause

- Mechanical Tissue Modeling II
  - Finite Element Method
  - Examples

Group work

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Motivation and Background

Finite Element Method allows solving of physical field problems taking
• anisotropy
• inhomogeneity
• nonlinearity
of material properties into account

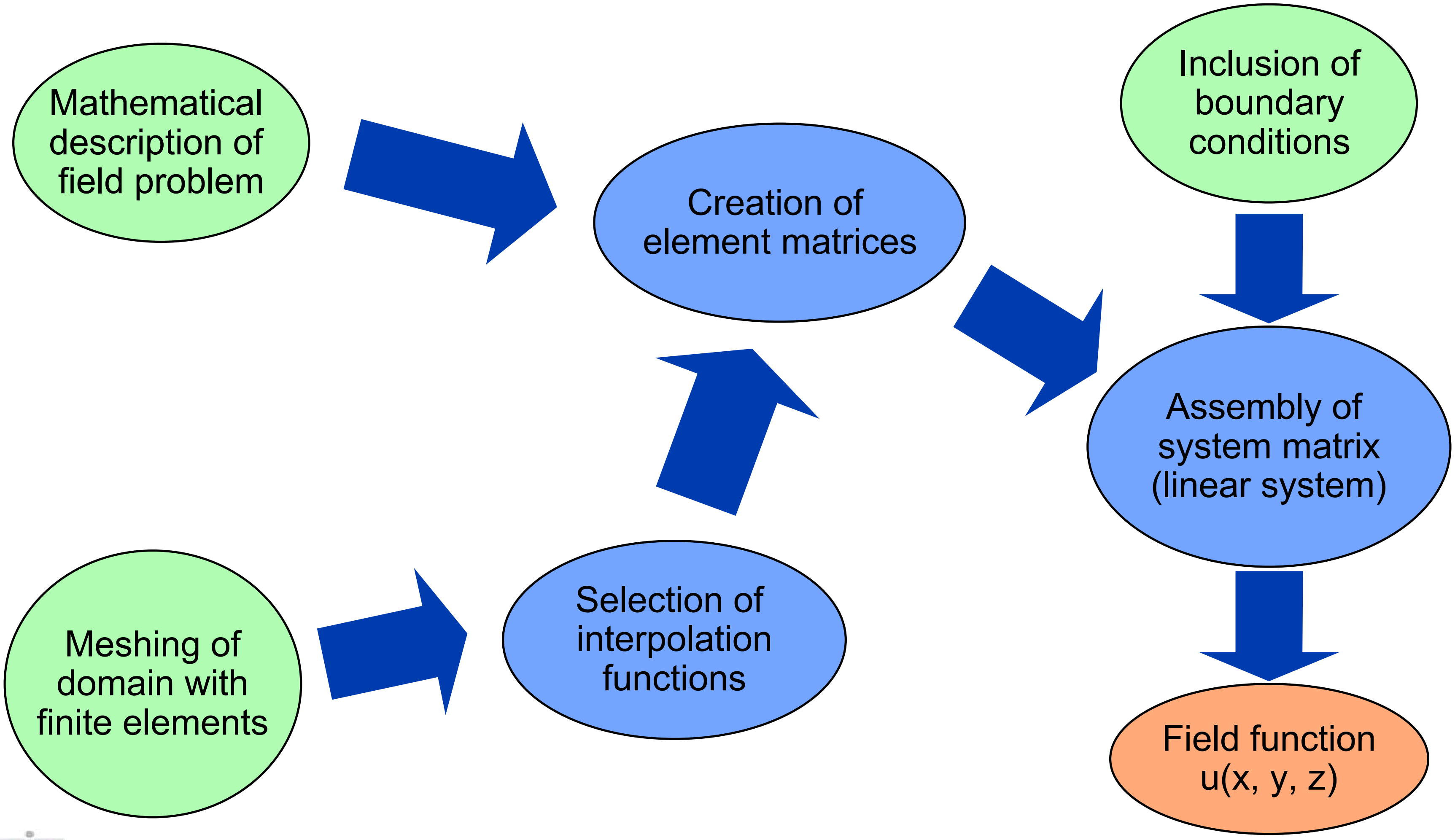
Applications

• Electrostatics
• (Quasi-)stationary electrical fields
• Wave propagation
• Temperature
• Structure and fluid mechanics
• …

Commercial packages

• Ansys
• EMAS
• ...

Finite Element Method: Overview



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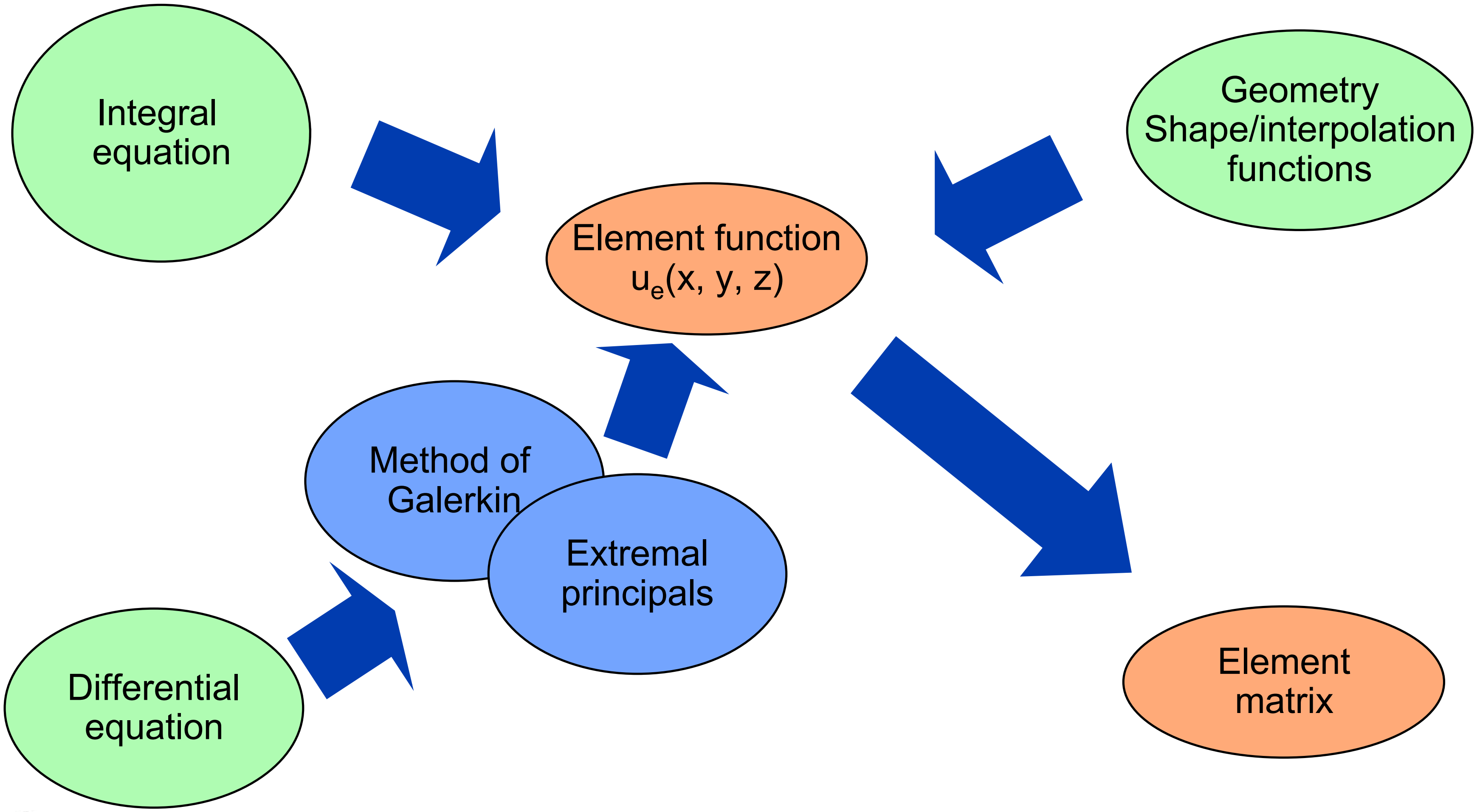
Finite Element Method: Element Matrix



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Direct Method: Integral Equations

$$W_e = \int_V \frac{1}{2} \varepsilon E^2 dV$$

W_e: Electrical energy

E: Electrical field

ε: Permittivity

$$P_L = \int_V \sigma E^2 \, dV$$

P_L: Electrical power

E: Electrical field

σ: Electrical conductivity

$$W_{elast} = \int_V \frac{1}{2} E \varepsilon^2 dV$$

W_{elast}: Elastic potential energy

E: Young's modulus

ε: Strain

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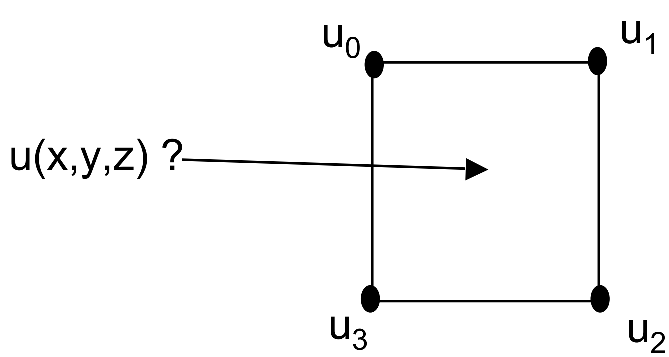
Shape Functions: Motivation

- Field values are given only at some points
- Calculation of surface/volume integrals necessitates interpolation of values in domain
- Element geometry is described with some points
- Coordinate transforms
  Local coordinates ⟷ Global coordinates



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Shape Functions

$$u(\vec{x}) = \sum_{k=0}^{K-1} u_k N_k(\vec{x})$$

u: Interpolation function

$\vec{x}$: Position vector

u_k: Field value at node k

N_k: Shape function

Commonly, interpolation is based on polynomial shape functions.

In general, shape functions fulfill:

$$N_k(\vec{x}) = \begin{cases} 1 & \text{at node } k \\ 0 & \text{at node } i \neq k \end{cases}$$

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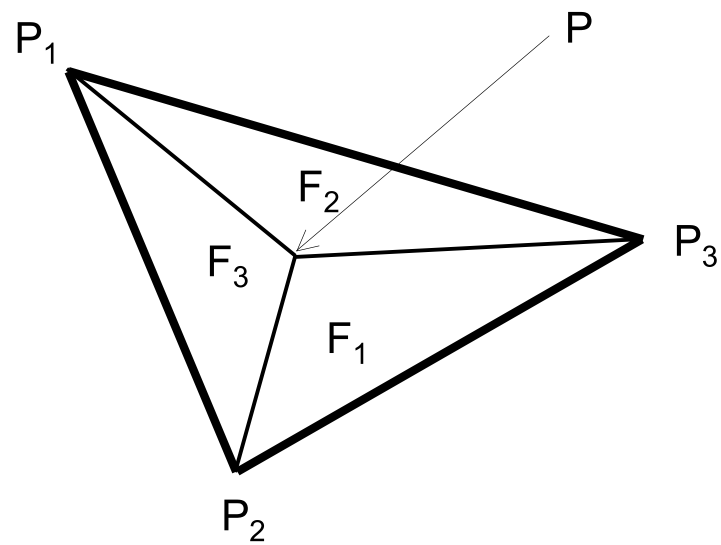
Shape Functions: Barycentric Coordinates in Triangle

$$N_i(\vec{x}) = \frac{F_i(\vec{x})}{\sum_{j=1}^{3} F_j(\vec{x})}$$



$$N_1(\vec{x}) + N_2(\vec{x}) + N_3(\vec{x}) = 1$$

$$N_k(\vec{x}) = \begin{cases} 1 & \text{at node } k \\ 0 & \text{at node } i \neq k \end{cases}$$

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Shape Functions 1D: Linear Lagrangian Interpolation

$$u(x) = a + bx$$

$$N_0(x) = (1 - x)$$

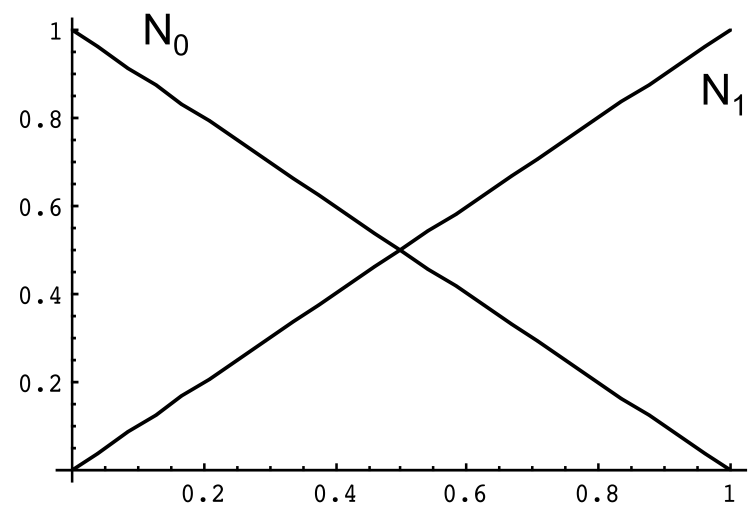
$$N_1(x) = x$$

$$u(x) = (1 - x)u_0 + xu_1$$

u: Field value function with parameter position

x: Position

u_0, u_1: Field value at node 0 and 1, respectively



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Example for Calculation of Shape Functions

General linear interpolation in 1D:

$$u(x) = a + bx$$

Node conditions:

$$u(0) = a + bx = u_0 \quad \Rightarrow \quad a = u_0$$
$$u(1) = a + bx = u_1 \quad \Rightarrow \quad a + b = u_1$$

Linear system of equation, inversion:

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}: \quad A\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} a \\ b \end{pmatrix} = A^{-1}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \text{ mit } A^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

Linear interpolation based on shape functions:

$$u(x) = u_0 + (-u_0 + u_1)\, x = (1 - x)\, u_0 + x\, u_1$$

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Shape Function 1D: Quadratic Lagrangian Interpolation

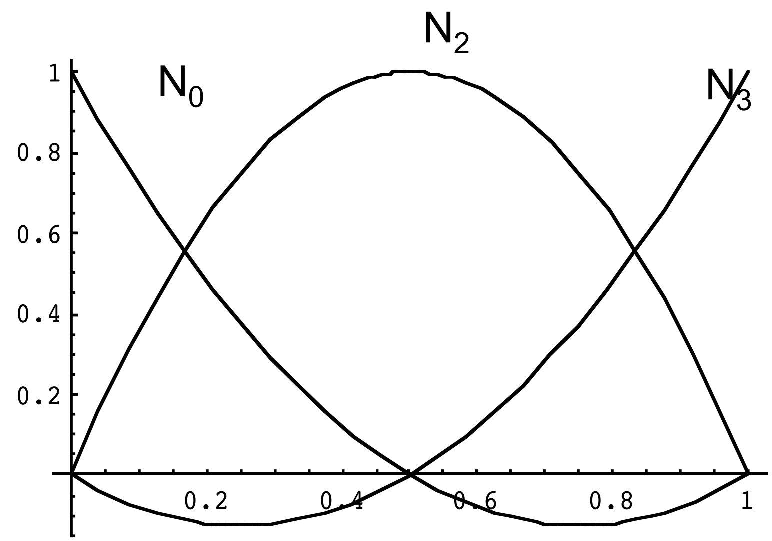
$$u(x) = a + bx + cx^2$$

$$N_0(x) = (1 - x)(1 - 2x)$$

$$N_1(x) = 4x(1 - x)$$

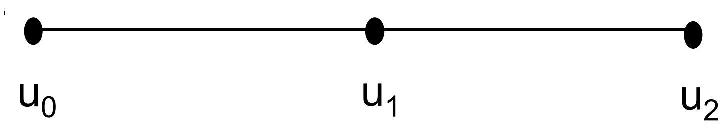
$$N_2(x) = -x(1 - 2x)$$

$$u(x) = N_0(x)u_0 + N_1(x)u_1 + N_2(x)u_2$$



x: Position

u_0, u_1, u_2: Field value at node 0, 1, and 2, respectively



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Shape Function 1D: Cubic Hermitian Interpolation

$$u(x) = a + bx + cx^2 + dx^3$$

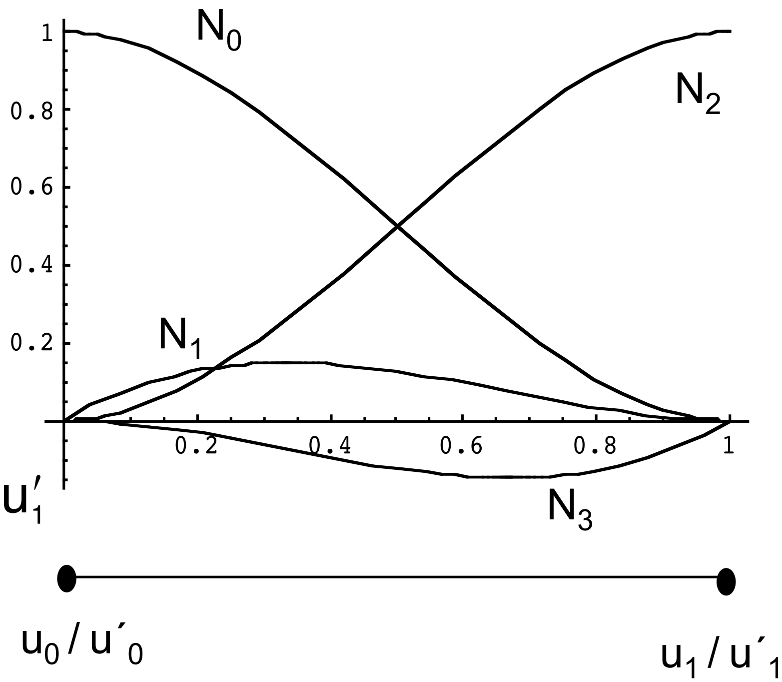
$$N_0(x) = (1-x)^2(1+2x)$$

$$N_1(x) = x(1-x)^2$$

$$N_2(x) = x^2(3-2x)$$

$$N_3(x) = -x^2(1-x)$$

$$u(x) = N_0(x)u_0 + N_1(x)u_0' + N_2(x)u_1 + N_3(x)u_1'$$



x: Position

u_0, u_1: Field value at node 0 and 1, respectively

u_0', u_1': Field derivative at node 0 and 1, respectively

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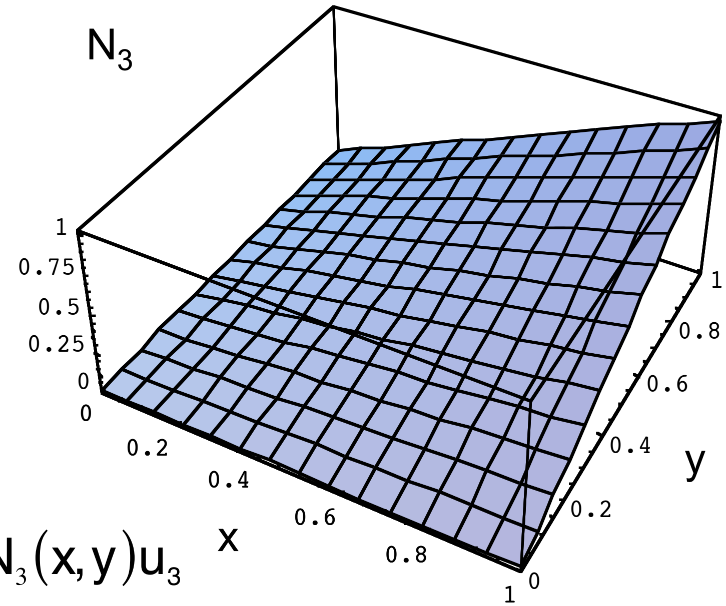
Shape Function 2D in Quad: Bilinear Lagrangian Interpolation

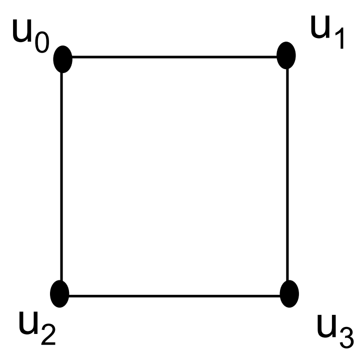
$$u(x,y) = a + bx + cy + dxy$$

$$N_0(x,y) = (1-x)(1-y)$$
$$N_1(x,y) = x(1-y)$$
$$N_2(x,y) = (1-x)y$$
$$N_3(x,y) = xy$$
$$u(x,y) = N_0(x,y)u_0 + N_1(x,y)u_1 + N_2(x,y)u_2 + N_3(x,y)u_3$$

x, y: Position

u_0, u_1, u_2, u_3: Field value at node 0..3





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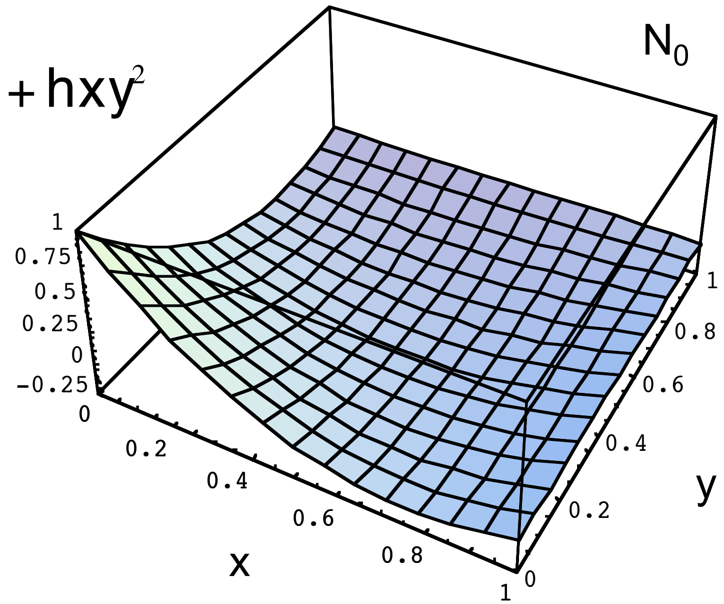
Shape Function 2D in Quad: Biquadratic Lagrangian Interpolation

$$u(x,y) = a + bx + cy + dxy + ex^2 + fy^2 + gx^2y + hxy^2$$



$$N_0(x,y) = (1-x)(1-y)(1-2x-2y)$$

$$N_1(x,y) = -x(1-y)(1-2x+2y)$$

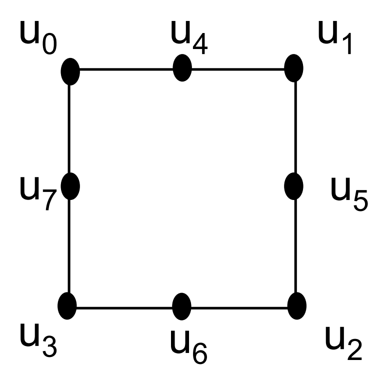
$$N_2(x,y) = xy(3-2x-2y)$$

...

$$u(x,y) = \sum_{k=0}^{7} N_k(x,y)\,u_k$$

x, y: Position

$u_0 - u_7$: Field value at node 0..7



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Shape Function 3D in Hexahedron: Trilinear Lagrangian Interpolation

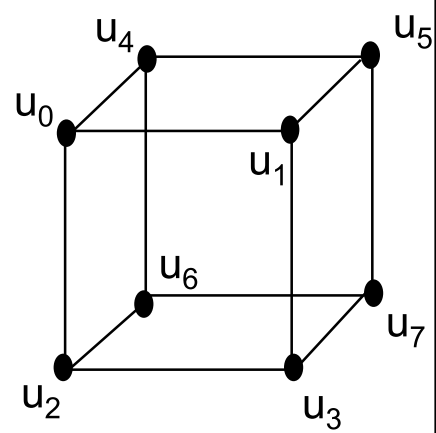
$$u(x,y,z) = a + bx + cy + dz + exy + fyz + gxz + hxyz$$

$$N_0(x,y,z) = (1-x)(1-y)(1-z)$$

$$\vdots$$

$$N_7(x,y,z) = xyz$$

$$u(x,y,z) = \sum_{k=0}^{7} N_k(x,y,z)\,u_k$$



x,y,z: Position

$u_0 - u_7$: Field value at node 0..7

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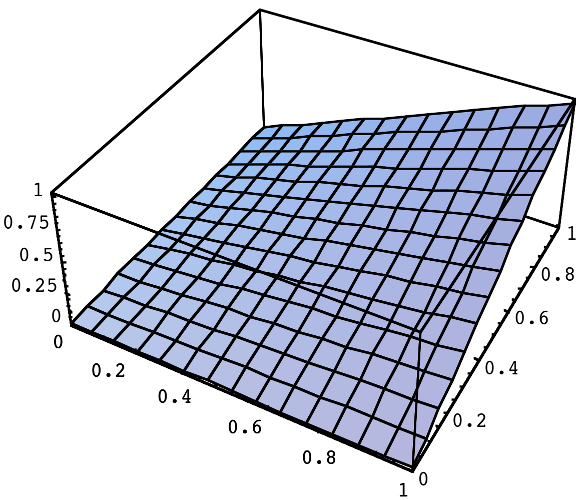
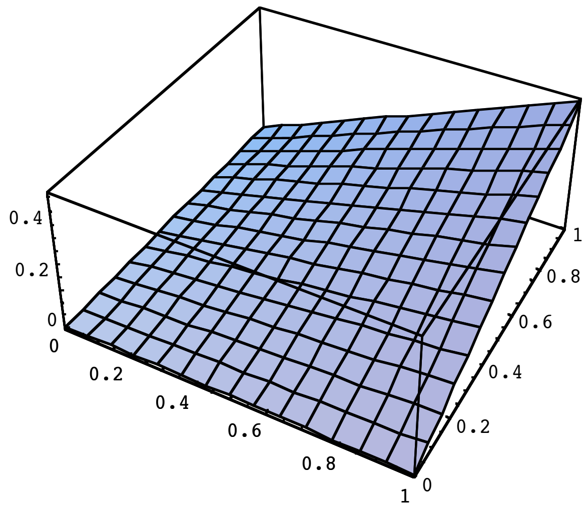
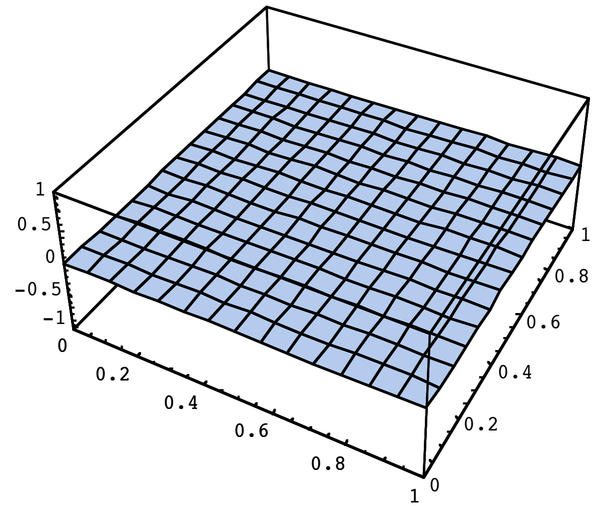
Shape Function 3D in Hexahedron: Trilinear Lagrangian Interpolation



z=1



z=.5



z=0

Unit cube: $N_7(x, y, z) = x\,y\,z$

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System Matrix and Vector of Stationary Problems

Transform of integral to system of linear equations

$$I = \int_G f(u(\vec{x}))\, dV \quad \Rightarrow \quad I = \vec{u}^T S\, \vec{u} + \vec{b}^T \vec{u} + c$$

f: Energy density function

u: Field function

$\vec{x}$: Position vector

$\vec{u}$: Field values/derivatives at nodes

S : System matrix

$\vec{b}$: System vector

c: Constant

Stationarity: $\forall_k \quad \frac{\partial I}{\partial u_k} = 0 \quad \Rightarrow \quad S\, \vec{u} + \vec{b} = 0$

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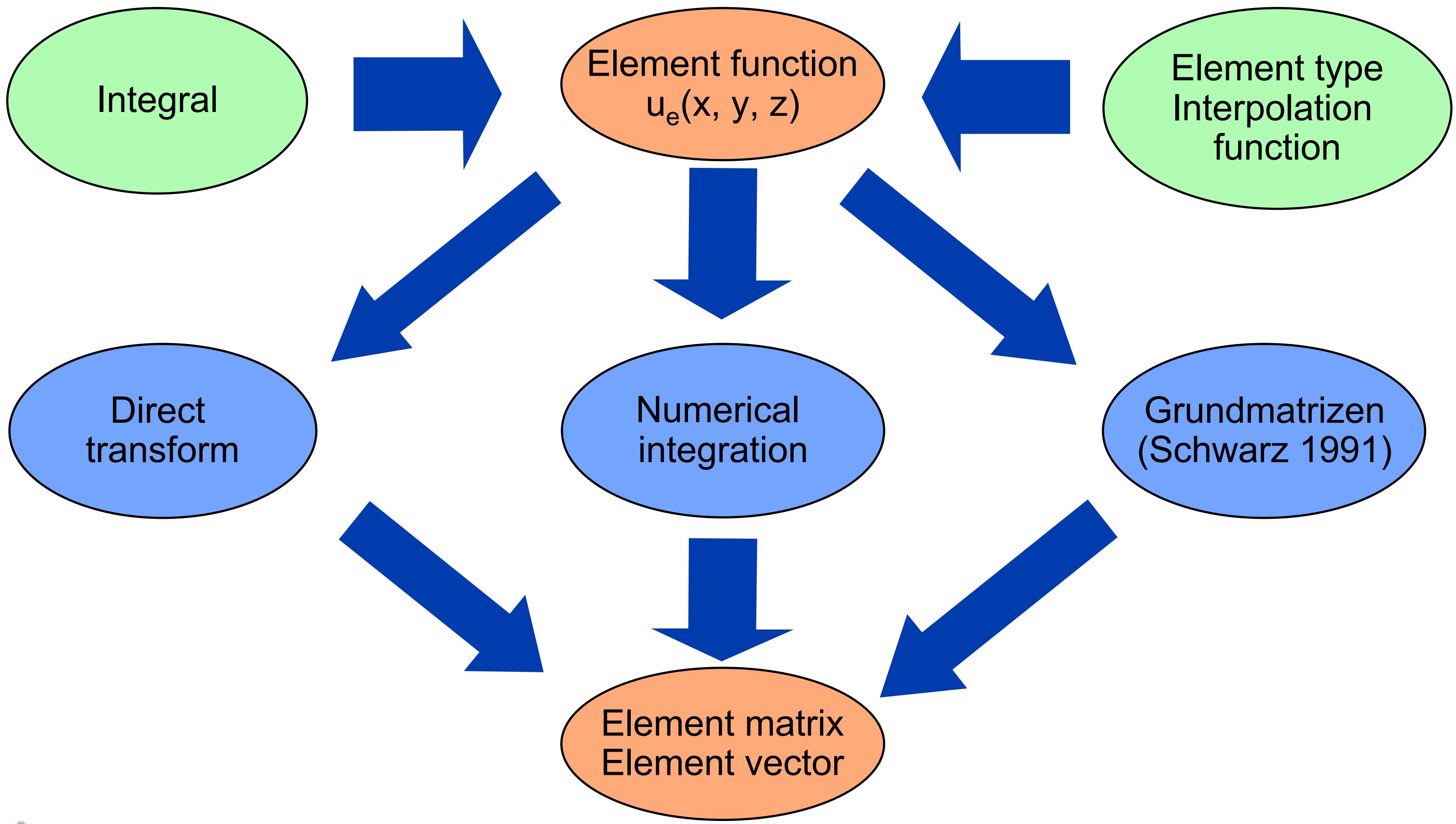
Element Matrix and Vector: Overview



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Element Matrix and Vector of Stationary Problems

Element-wise transform of integral to system of linear equations

$$I_e = \int_E f\left(u(\vec{x})\right)\, dV \quad \Rightarrow \quad I_e = \vec{u}_e^{\,T} S_e \vec{u}_e + \vec{b}_e^{\,T} \vec{u}_e + c$$

f: Energy density function

u: Field function

$\vec{x}$: Position vector

$\vec{u}_e$: Field values/derivatives at nodes

S_e: System matrix

$\vec{b}_e$: System vector

c: Constant

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Transform of Element Function with Shape Functions

$$I_e = \int_E f\left(u(\vec{x})\right) dV \quad \overset{u(\vec{x}) = \sum_{k=0}^{K-1} u_k N_k(\vec{x})}{\Longrightarrow} \quad I_e = \int_E \sum_{k=0}^{K-1} f\left(u_k N_k(\vec{x})\right) dV$$

u_k: Field variable at node k

N_k: Shape function

$$I_e = \int_E \sum_{k=0}^{K-1} f\left(u_k N_k(\vec{x})\right) dV \quad \overset{\text{Direct Transform / Numerical Integration}}{\Longrightarrow} \quad I_e = \vec{u}_e^{\,T} S_e \vec{u}_e + \vec{b}_e^{\,T} \vec{u}_e + c$$

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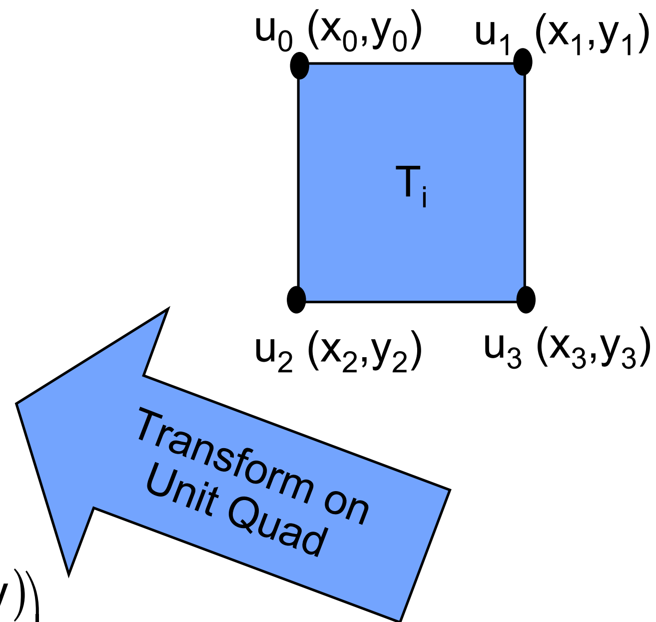
Direct Transform of Element Function

Quad, bilinear lagrangian



$$u(x,y) = a + bx + cy + dxy$$

$$N_0(x,y) = (1-x)(1-y),\quad N_1(x,y) = x(1-y),$$

$$N_2(x,y) = (1-x)y,\quad N_3(x,y) = xy$$

$$u(x,y) = N_0(x,y)u_0 + N_1(x,y)u_1 + N_2(x,y)u_2 + N_3(x,y)u_3$$

$$\iint_{T_i} u(x,y)\,dx\,dy = \vec{u}_e^T \iint_{T_i} \vec{N}(x,y)\,dx\,dy = \vec{u}_e^T J \int_0^1\!\!\int_0^1 \vec{N}(x,y)\,dx\,dy$$

J: Jacobian

Matrix representation:

$$\iint_{T_i} u(x,y)\,dx\,dy = = \begin{pmatrix} u_0 \\ \vdots \\ u_3 \end{pmatrix}^T \begin{vmatrix} x_0 - x_2 & x_1 - x_2 \\ y_0 - y_2 & y_1 - y_2 \end{vmatrix} \int_0^1\!\!\int_0^1 \begin{pmatrix} (1-x)(1-y) \\ \vdots \\ xy \end{pmatrix} dx\,dy$$

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Element Vector for Bilinear Lagrangian Quad: Example

$$I = \iint_{T_i} u(x,y)\, dx\, dy = \begin{pmatrix} u_0 \\ \vdots \\ u_3 \end{pmatrix}^T \begin{vmatrix} x_0 - x_2 & x_1 - x_2 \\ y_0 - y_2 & y_1 - y_2 \end{vmatrix} \int_0^1 \int_0^1 \begin{pmatrix} (1-x)(1-y) \\ \vdots \\ x\,y \end{pmatrix} dx\, dy$$

$$\begin{pmatrix} \frac{\partial I}{\partial u_0} \\ \frac{\partial I}{\partial u_1} \\ \frac{\partial I}{\partial u_2} \\ \frac{\partial I}{\partial u_3} \end{pmatrix} = \begin{vmatrix} x_0 - x_2 & x_1 - x_2 \\ y_0 - y_2 & y_1 - y_2 \end{vmatrix} \begin{pmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{pmatrix} = \vec{b}_e$$

Element vector

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Element Matrices From Grundmatrizen

Integrals:

$$\int_0^l u^2(x)dx, \quad \int_0^l u'(x)^2 dx, \quad \int_0^l u''(x)^2 dx, \quad \int_0^l u(x)dx$$

l: Length

u: Field function

Transform to unit interval [0,1] by

$x = l\,\xi$

gives Grundmatrizen with geometry factor

$$l\int_0^1 u^2(\xi)d\xi, \quad \frac{1}{l}\int_0^1 u'(\xi)^2 d\xi, \quad \frac{1}{l^2}\int_0^1 u''(\xi)^2 d\xi, \quad l\int_0^1 u(\xi)d\xi$$

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Group Work

The finite element method allows to solve field problems based on a discretization of the spatial domain and interpolation of field values.

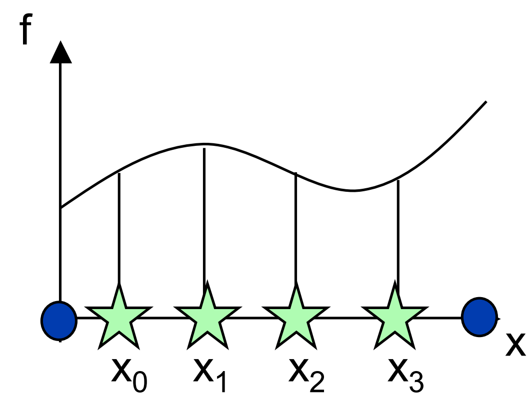
How will
• shape functions
• element types
influence the accuracy of a solution?

Numerical Integration of Element Functions

Calculation of element matrices and vectors can be complicated!

Instead: Numerical integration, e.g. with Gaussian quadrature

$$\int_V f(\vec{x})\, d\vec{x} = \sum_{i=0}^{I-1} W_i f(x_i) + E$$



W_i: Integration weights

E: Approximation error

$\vec{x}$: Position

x_i: Integration points ☆

u_k: Node points ●

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Gaussian Quadrature: 1D Example

Approximation of integrals i with cubic function f

$$f(x) = a + bx + cx^2 + dx^3$$

$$I = \int_0^1 f(x)\,dx = a\int_0^1 dx + b\int_0^1 x\,dx + c\int_0^1 x^2 dx + d\int_0^1 x^3\,dx$$

$$I = W_0 f(x_0) + W_1 f(x_1)$$

Calculation of weights and points by selecting appropriate parameters a, b, c, and d.

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Gaussian Quadrature: Parameters

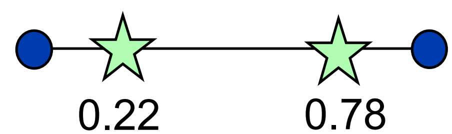
$$a = 1, b = c = d = 0: \quad \int_0^1 dx = 1 = W_0 + W_1$$

$$b = 1, a = c = d = 0: \quad \int_0^1 x\,dx = \frac{1}{2} = W_0 x_0 + W_1 x_1$$

$$c = 1, a = b = d = 0: \quad \int_0^1 x^2\,dx = \frac{1}{3} = W_0 x_0^{\,2} + W_1 x_1^{\,2}$$

$$d = 1, a = b = c = 0: \quad \int_0^1 x^3\,dx = \frac{1}{4} = W_0 x_0^{\,3} + W_1 x_1^{\,3}$$

$$\Rightarrow W_0 = W_1 = \frac{1}{2}, \quad x_{0/1} = \frac{1}{2} \pm \frac{1}{2\sqrt{3}}$$



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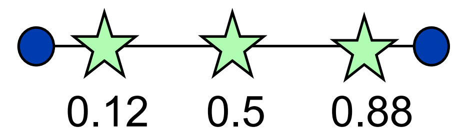
Gaussian Quadrature: Examples

$$f(x) = a + b\,x + c\,x^2 + d\,x^3 + e\,x^4 + f\,x^5$$

$$\Rightarrow W_0 = W_2 = \frac{5}{18}, \quad W_1 = \frac{4}{9}$$

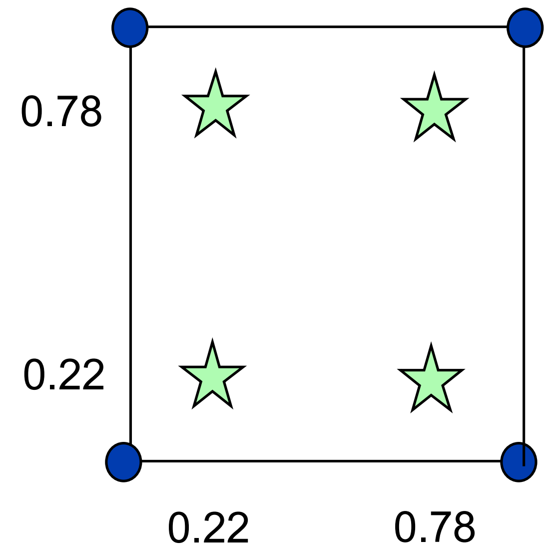
$$x_{0/2} = \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{3}{5}}, \quad x_1 = \frac{1}{2}$$



$$f(x,y) = a + b\,x + c\,y + d\,xy$$

$$\Rightarrow W_0 = W_1 = W_2 = W_3 = \frac{1}{4}$$

$$x_{0/1} = y_{0/1} = \frac{1}{2} \pm \frac{1}{2\sqrt{3}}$$



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Order of Gaussian Quadrature and Degree of Approximation

Dimension	Domain	Points	Approximation
1	Curve	1	1st order polynomial
1	Curve	2	3rd order polynomial
1	Curve	3	5th order polynomial
1	Curve	n	2n-1. order polynomial
2	Quad	1 x 1	
2	Quad	2 x 2	
2	Quad	n x n	
3	Hexahedron	1 x 1 x 1	
3	Hexahedron	2 x 2 x 2	
3	Hexahedron	n x n x n	

Evaluation of Integration Methods

Direct method for integration
- only for element functions of low complexity
- mathematically demanding
- efficiency depends on complexity of element function

Grundmatrizen method
- only for element functions and types in library
- simple implementation
- efficient

Integration with Gaussian quadrature
- many element functions and types
- mathematically “trivial”
- efficient

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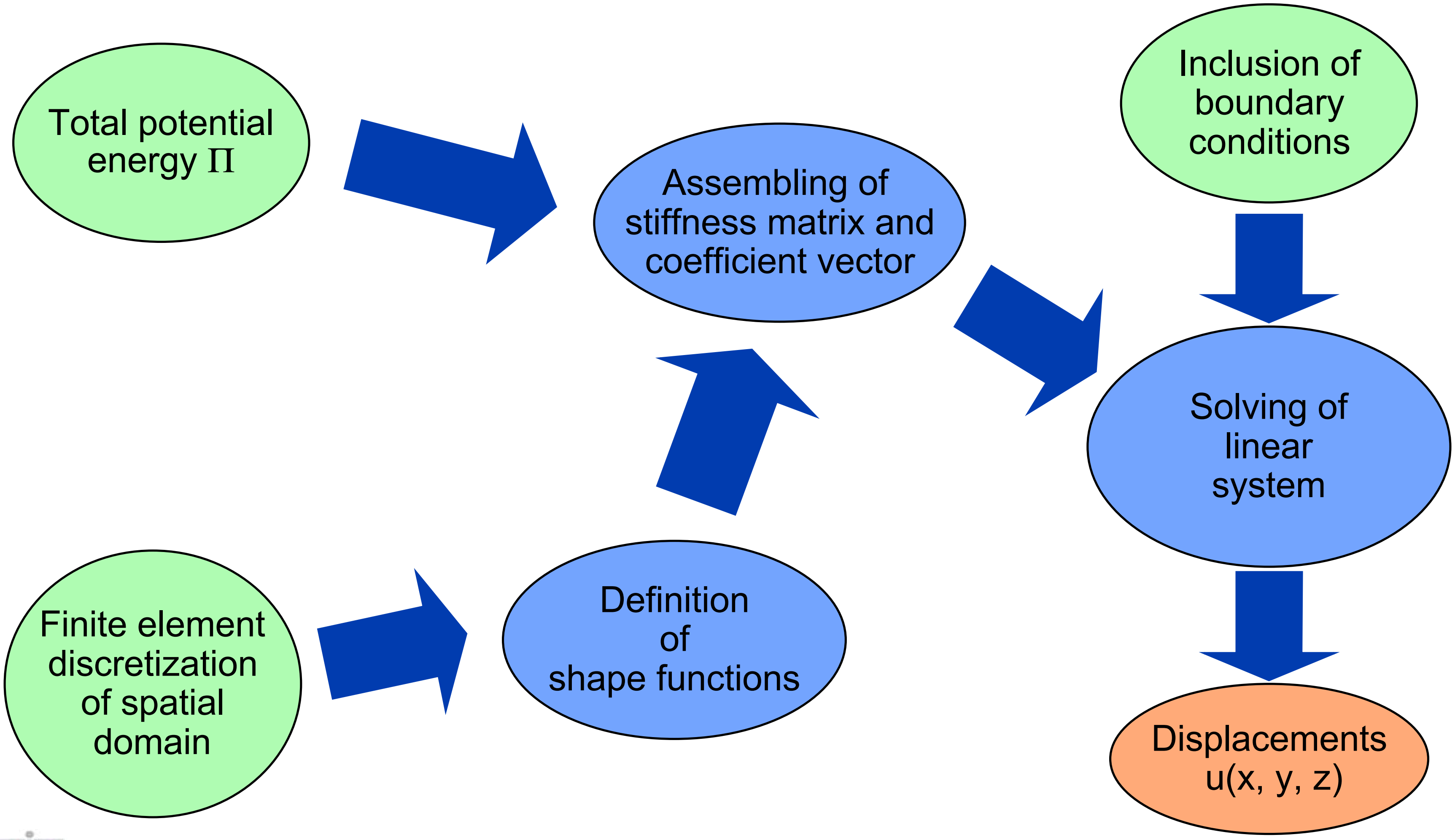
Finite Element Method for Static Linear Elasticity



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Static Linear Elasticity Problems: Finite Element Method

$$\Pi = \frac{1}{2}\iiint_V \hat{\sigma}^T \hat{\varepsilon}\, dV - \iiint_V p^T u\, dV - \iint_S q^T u\, dS - \sum_i F_i^T u_i$$

Π: Total potential energy

u / u_i: Displacement vector

$\hat{\sigma}$: Cauchy stress vector

$\hat{\varepsilon}$: Classical strain vector

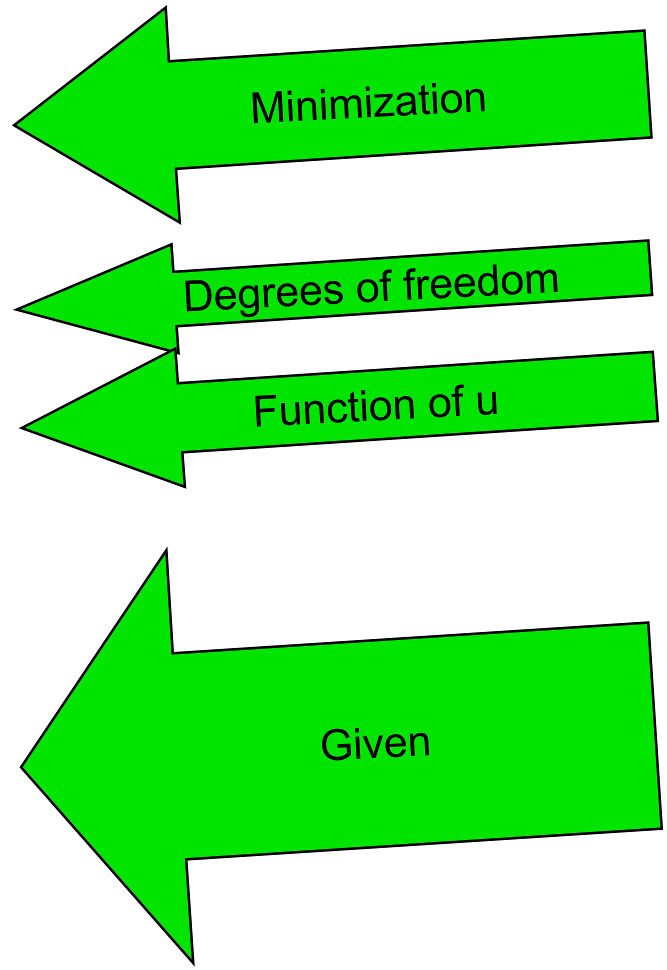
p: Force in volume

q: Force on surface

F_i: Single force in volume

V: Spatial domain

S: Surface



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Static Linear Elasticity Problems

$$u = \begin{pmatrix} u_x & u_y & u_z \end{pmatrix}^T$$

$$\hat{\varepsilon} = \begin{pmatrix} \varepsilon_x & \varepsilon_y & \varepsilon_z & \varepsilon_{xy} & \varepsilon_{yz} & \varepsilon_{xz} \end{pmatrix}^T$$

$$= \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_z}{\partial z} & \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} & \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \end{pmatrix}^T$$

$$= \begin{pmatrix} \frac{1}{E}\left(\sigma_x - \nu\sigma_y - \nu\sigma_y\right) & \cdots\cdots\cdot & \frac{2(1+\nu)}{E}\sigma_{xy} & \cdots\cdots\cdot \end{pmatrix}^T$$

$$= C^{-1}\hat{\sigma}$$

$$\hat{\sigma} = \begin{pmatrix} \sigma_x & \sigma_y & \sigma_z & \sigma_{xy} & \sigma_{yz} & \sigma_{xz} \end{pmatrix}^T$$

C: Material matrix

E: Young’s modulus

ν: Poisson’s ratio

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Static Linear Elasticity Problems: Material Matrix

$$\hat{\varepsilon} = C^{-1}\hat{\sigma} \quad \overset{\text{Inversion}}{\Rightarrow} \quad \hat{\sigma} = C\hat{\varepsilon}$$

$$C = \frac{E}{(1+\nu)(1-2\nu)} \begin{pmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) \end{pmatrix}$$

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Finite Element Formulation with Shape Functions I

$$\Pi = \frac{1}{2}\iiint_V \hat{\sigma}^T \hat{\varepsilon}\, dV - \iiint_V p^T u\, dV - \iint_S q^T u\, dS$$

Shape functions: $N = \left(N_1(x,y,z) \quad \cdots \quad N_k(x,y,z)\right)^T$

Node variables: $u_e = \left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdots \begin{pmatrix} x_k \\ y_k \\ z_k \end{pmatrix}\right)^T$

Relationship between strain and stress: $\hat{\sigma} = C\hat{\varepsilon}$

$$\Pi = \frac{1}{2}\iiint_V (C\hat{\varepsilon})^T \hat{\varepsilon}\, dV - \iiint_V p^T N^T u_e\, dV - \iint_S q^T N^T u_e\, dS$$

CVRTI

Finite Element Formulation with Shape Functions II

$$\Pi = \frac{1}{2}\iiint_V (C\hat{\varepsilon})^T \hat{\varepsilon}\, dV - \iiint_V p^T N^T u_e\, dV - \iint_S q^T N^T u_e\, dS$$

$\hat{\varepsilon} = B u_e$ B: Transfer operator created from derivatives of shape functions

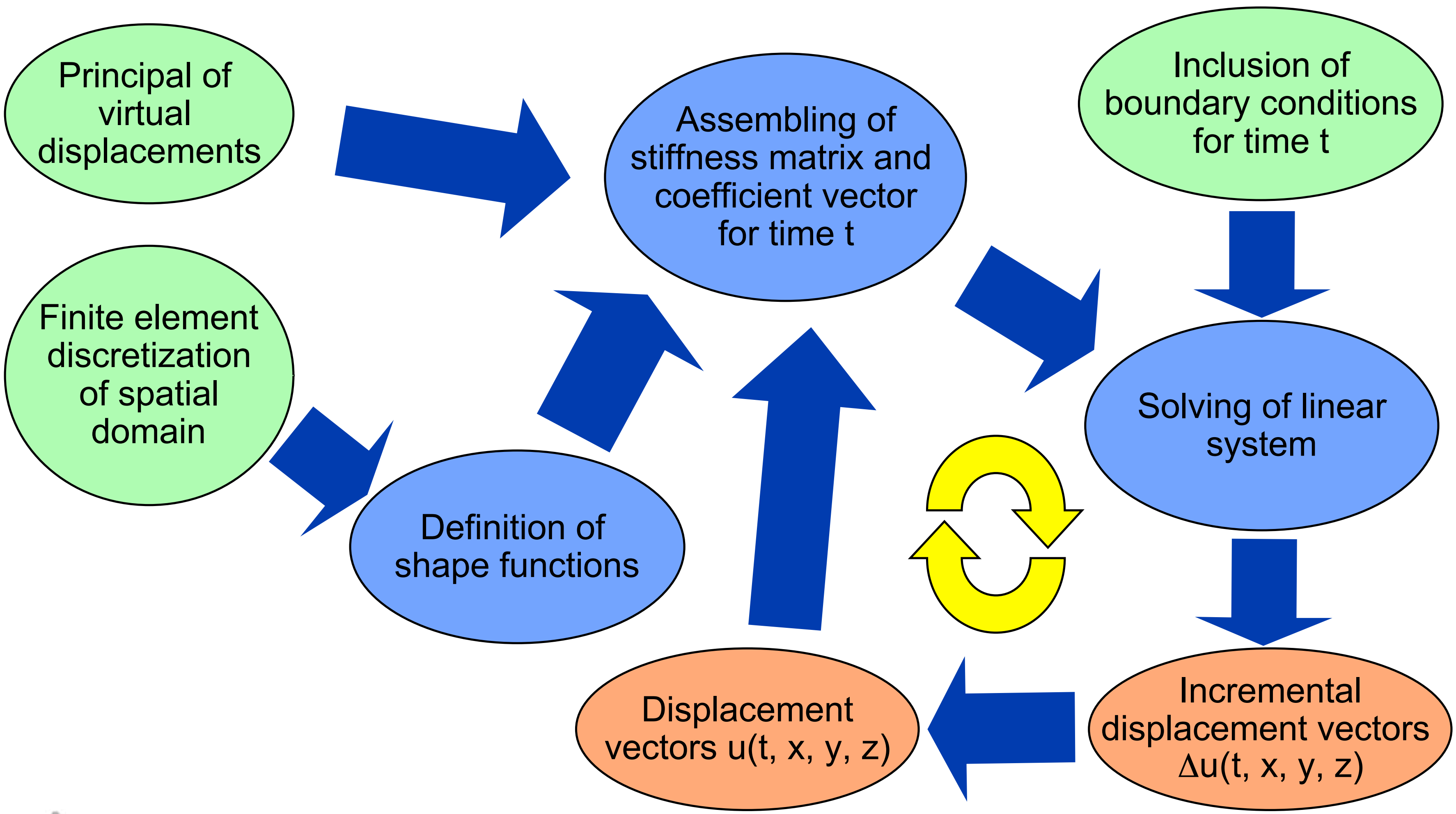
$$\Pi = \frac{1}{2}\iiint_V (C B u_e)^T B u_e\, dV - \iiint_V p^T N^T u_e\, dV - \iint_S q^T N^T u_e\, dS$$

$$= \frac{1}{2} u_e^T \iiint_V B^T C^T B\, dV\, u_e - \iiint_V p^T N^T\, dV\, u_e - \iint_S q^T N^T dS\, u_e$$

$$\Pi = \frac{1}{2} u_e^T S\, u_e - K u_e$$

S: Stiffness matrix   K: Coefficient vector

Lagrangian Incremental Formulation



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Principal of Virtual Displacements

"The equilibrium of the body requires that any small, virtual displacement leads to identical internal and external work"

Internal Work — External Work

$$\iiint_{^{t+\Delta t}V} {}^{t+\Delta t}\sigma_{ij}\, \delta\, {}_{t+\Delta t}\varepsilon_{ij}\, d\,{}^{t+\Delta t}V = \iiint_{^{t+\Delta t}V} {}^{t+\Delta t}p_i\, \delta u_i\, d\,{}^{t+\Delta t}V + \iint_{^{t+\Delta t}S} {}^{t+\Delta t}q_i\, \delta u_i\, d\,{}^{t+\Delta t}S$$

Summation Convention!

σ: Cauchy stress tensor

$\delta\varepsilon$: Variation of infinitesimal strain tensor

δu: Virtual displacement

p,q: Volume and surface force vector, resp.

Bathe: Finite Element Procedures, Page 156

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Incremental Lagrangian Formulation

$$\iiint_{^{t+\Delta t}V} {}^{t+\Delta t}\sigma_{ij}\,\delta\,{}_{t+\Delta t}\varepsilon_{ij}\; d\,{}^{t+\Delta t}V = \iiint_{^{t+\Delta t}V} {}^{t+\Delta t}p_i\,\delta u_i\; d\,{}^{t+\Delta t}V + \iint_{^{t+\Delta t}S} {}^{t+\Delta t}q_i\,\delta u_i\; d\,{}^{t+\Delta t}S = {}^{t+\Delta t}R$$

σ: Cauchy stress tensor

$\delta\varepsilon$: Variation of infinitesimal strain tensor

δu: Virtual displacement

p,q: Volume and surface force vector, resp.

R: Load vector

$$\iiint_{^{0}V} {}^{t+\Delta t}_{\;0}S_{ij}\,\delta\,{}^{t+\Delta t}_{\;0}E_{ij}\; d\,{}^{0}V = {}^{t+\Delta t}R$$

Summation convention!

S: 2. Piola-Kirchhoff stress tensor

δE: Variation of Lagrangian tensor

Bathe: Finite Element Procedures, page 522

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Incremental Lagrangian Formulation: Given / Wanted

$$\iiint_{^0V} {}^{t+\Delta t}_{\;\;0}S_{ij}\,\delta\,{}^{t+\Delta t}_{\;\;0}E_{ij}\;d^0V = \iiint_{^0V} {}^{t+\Delta t}_{\;\;0}p_i\,\delta u_i\;d^0V + \iint_{^0S} {}^{t+\Delta t}_{\;\;0}q_i\,\delta u_i\;d^0S$$

S:	2. Piola-Kirchhoff stress tensor	← Function von u
δE:	Variation of Lagrangian strain tensor	← Function von u
δu:	Variation of displacement	← Wanted for $t+\Delta t$
u:	Displacement	← Given for t
p,q:	Volume and surface force vector, resp.	← Given for t

CVRTI

Linearization of Internal Work

$$ {}^{t+\Delta t}_{\ 0}S_{ij}\,\delta\,{}^{t+\Delta t}_{\ 0}E_{ij} = {}^{t}_{0}S_{ij}\,\delta\,{}^{t}_{0}E_{ij} + \frac{\partial}{\partial\, {}^{t}u_k}\left({}^{t}_{0}S_{ij}\,\delta\,{}^{t}_{0}E_{ij}\right) du_k $$

u: Displacement/node variable

Chain rule:

$$ \frac{\partial}{\partial\, {}^{t}u_k}\left({}^{t}_{0}S_{ij}\,\delta\,{}^{t}_{0}E_{ij}\right) du_k = \frac{\partial\, {}^{t}_{0}S_{ij}}{\partial\, {}^{t}u_k}\,\delta\,{}^{t}_{0}E_{ij}\, du_k + {}^{t}_{0}S_{ij}\,\frac{\partial\left(\delta\,{}^{t}_{0}E_{ij}\right)}{\partial\, {}^{t}u_k}\, du_k $$

$$ = \dots $$

$$ = {}_{0}C_{ijrs}\,\frac{\partial\, {}^{t}_{0}E_{rs}}{\partial\, {}^{t}u_k}\,\frac{\partial\, {}^{t}_{0}E_{ij}}{\partial\, {}^{t}u_l}\,\delta u_l\, du_k + {}^{t}_{0}S_{ij}\,\frac{\partial^2\, {}^{t}_{0}E_{ij}}{\partial\, {}^{t}u_k\,\partial\, {}^{t}u_l}\,\delta u_l\, du_k $$

with incremental stiffness matrix: $\quad {}_{0}C_{ijrs} = \dfrac{\partial\, {}^{t}_{0}S_{ij}}{\partial\, {}^{t}_{0}E_{rs}}$

Bathe: Finite Element Procedures, page 539

Element Stiffness Matrix

$$\iiint_{{}^0V} {}^{t+\Delta t}_{0}S_{ij}\,\delta\,{}^{t+\Delta t}_{0}E_{ij}\; d^0V = {}^{t+\Delta t}R$$

$$\iiint_{{}^0V} {}^{t}_{0}S_{ij}\delta{}^{t}_{0}E_{ij} + \frac{\partial}{\partial\,{}^tu_k}\left({}^{t}_{0}S_{ij}\delta{}^{t}_{0}E_{ij}\right)du_k\; d^0V = {}^{t+\Delta t}R$$

$$\iiint_{{}^0V} {}^{t}_{0}S_{ij}\delta{}^{t}_{0}E_{ij} + {}_0C_{ijrs}\frac{\partial{}^{t}_{0}E_{rs}}{\partial\,{}^tu_k}\frac{\partial{}^{t}_{0}E_{ij}}{\partial\,{}^tu_l}\delta u_l du_k + {}^{t}_{0}S_{ij}\frac{\partial^2\,{}^{t}_{0}E_{ij}}{\partial\,{}^tu_k\partial\,{}^tu_l}\delta u_l du_k\; d^0V = {}^{t+\Delta t}R$$

$$\left(\iiint_{{}^0V} {}_0C_{ijrs}\frac{\partial{}^{t}_{0}E_{rs}}{\partial\,{}^tu_k}\frac{\partial{}^{t}_{0}E_{ij}}{\partial\,{}^tu_l} + {}^tS_{ij}\frac{\partial^2\,{}^{t}_{0}E_{ij}}{\partial\,{}^tu_k\partial\,{}^tu_l}\, d^0V\right)du_k\delta u_l = {}^{t+\Delta t}R - \left(\iiint_{{}^0V} {}^tS_{ij}\frac{\partial{}^{t}_{0}E_{ij}}{\partial\,{}^tu_l}d^0V\right)\delta u_l$$

$$(K_L \quad + \quad K_{NL})\quad \Delta U \quad = \quad F$$

K_L: Linear stiffness matrix $\quad$ K_{NL}: Nonlinear stiffness matrix

ΔU: Incremental displacements $\quad$ F: Force vector

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Transfer Operator, Strain and Stress Vectors

Lagrangian strain vector calculated from displacements:

$$\hat{E} = \left(E_x \quad E_y \quad E_z \quad E_{xy} \quad E_{yz} \quad E_{xz}\right)^T = B\left(u_x \quad u_y \quad u_z\right)^T$$

$B = B(u)$: nonlinear transfer operator
decomposition in linear and nonlinear operator

2. Piola-Kirchhoff stress vector from derivative of strain energy density function:

$$\hat{S} = \left(S_x \quad S_y \quad S_z \quad S_{xy} \quad S_{yz} \quad S_{xz}\right)^T = \frac{\partial W}{\partial \hat{E}}$$

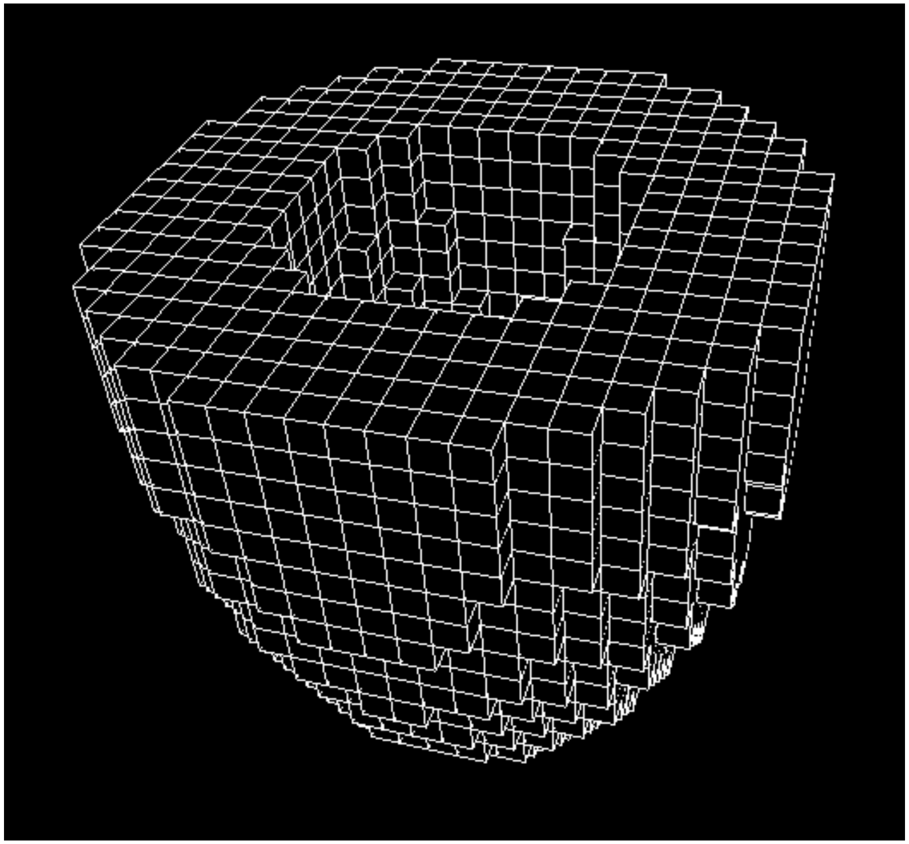
Incremental material matrix from derivative of stress tensor:

$$\hat{C} = \frac{\partial \hat{S}}{\partial \hat{E}}$$

Example: Passive Cardiac Mechanics

Left ventricle model
- approximated with 3752 cubic elements
- trilinear shape functions
- 3 versions of fiber orientation
- hyperelastic material (Guccione et al. 1991)
- incompressible

Boundary condition
- tension 1 kPa in fiber direction
- homogeneous



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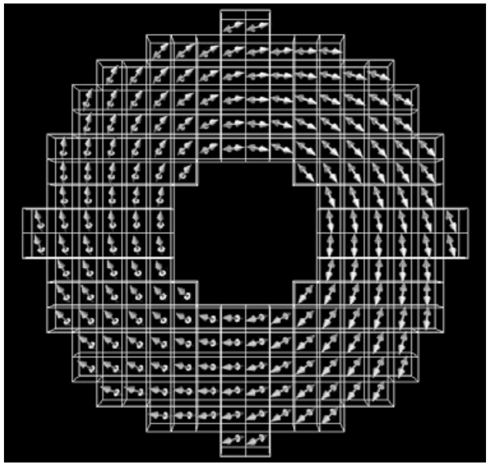
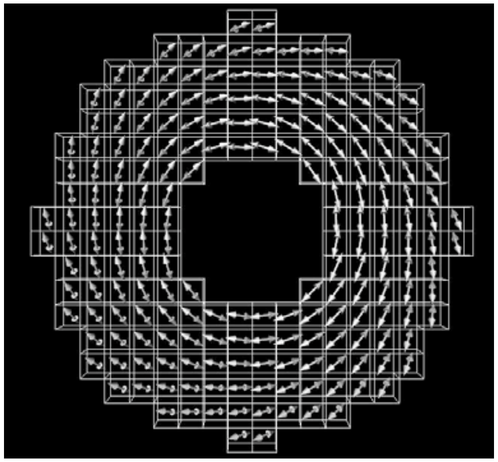
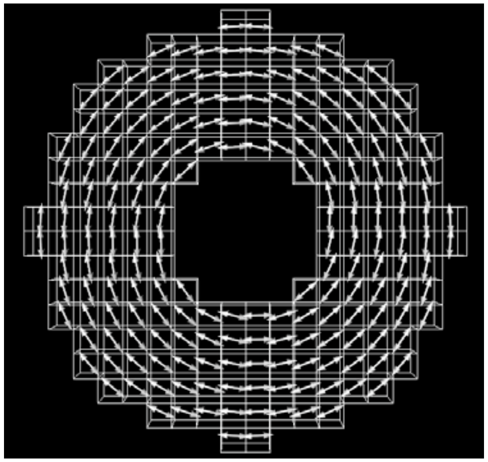
Example: Versions of Fiber Orientation



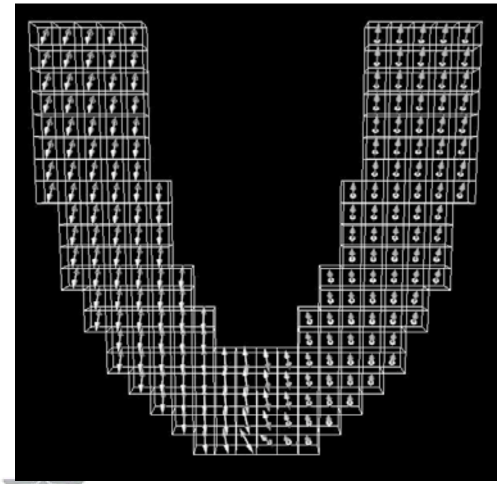
-45°, -45°, -45°

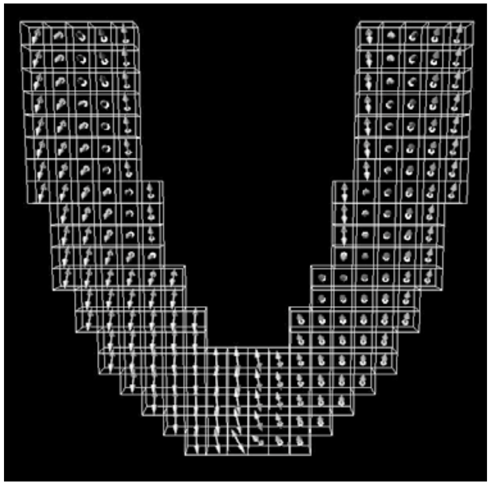


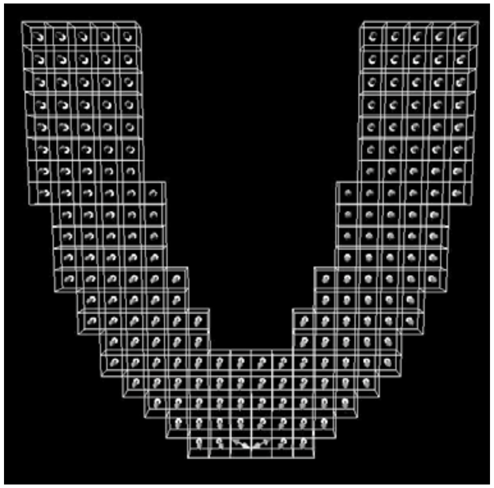
-45°, 0°, 45°



0°, 0°, 0°







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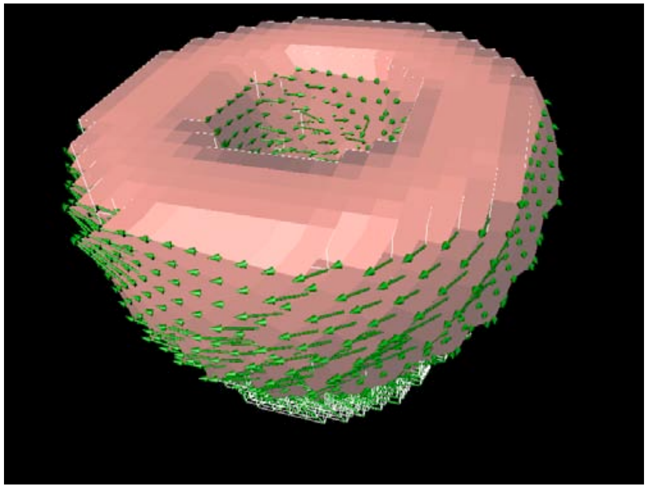
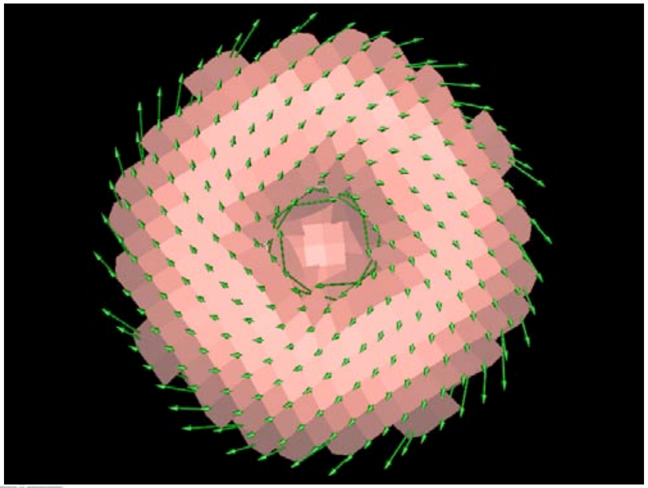
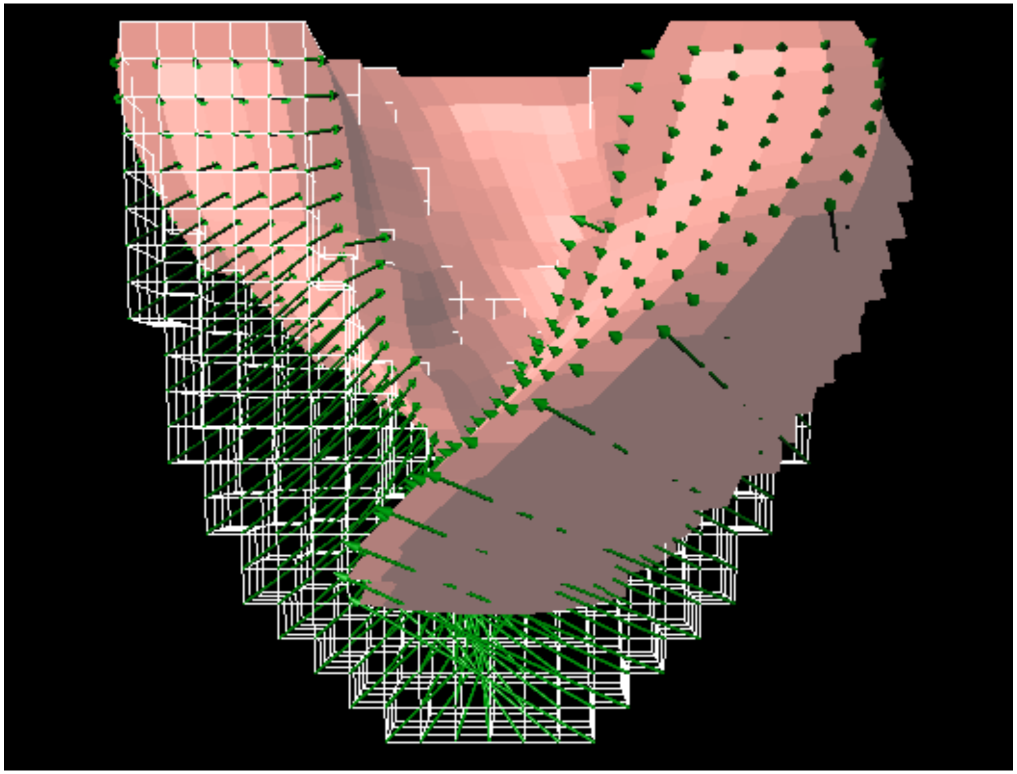
Example: -45°, -45°, -45°







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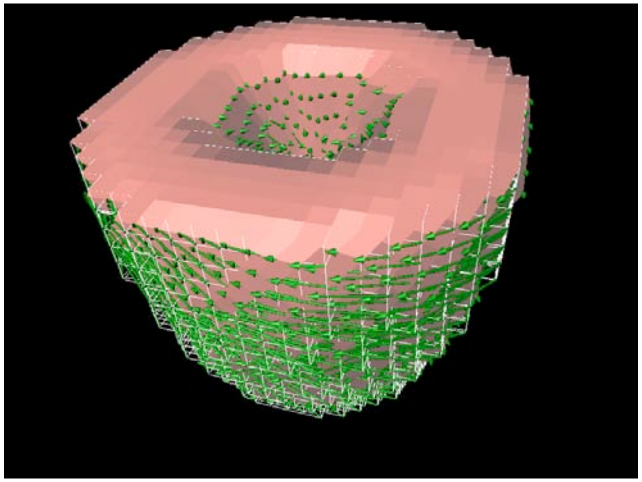
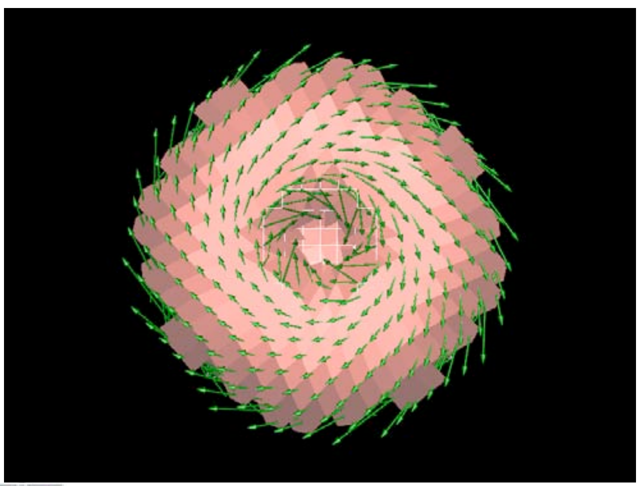
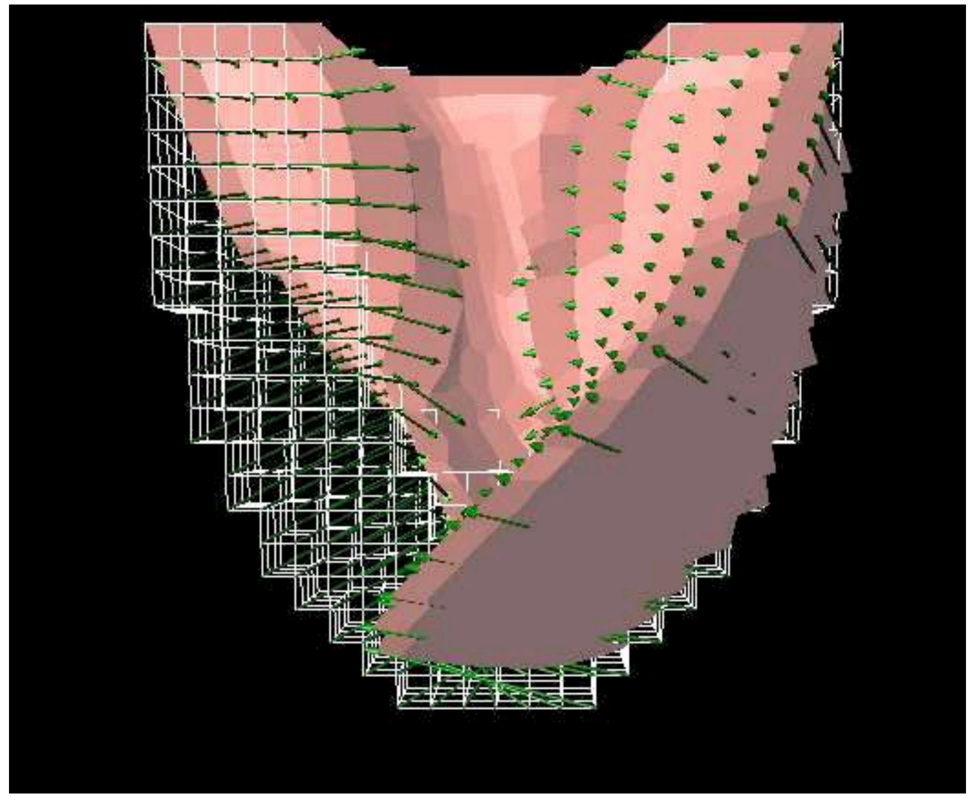
Example: -45°, 0°, 45°







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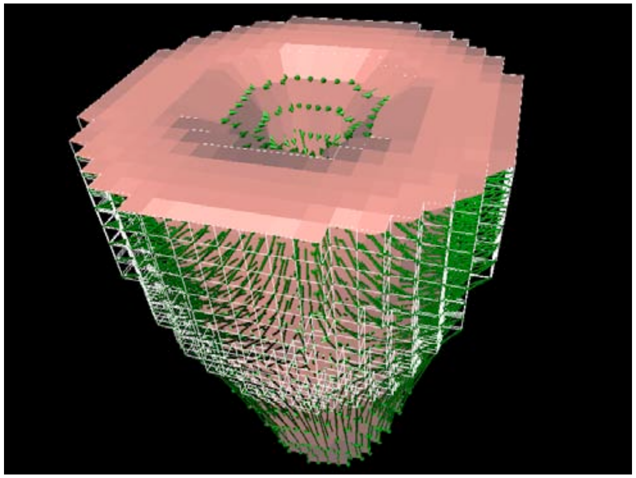
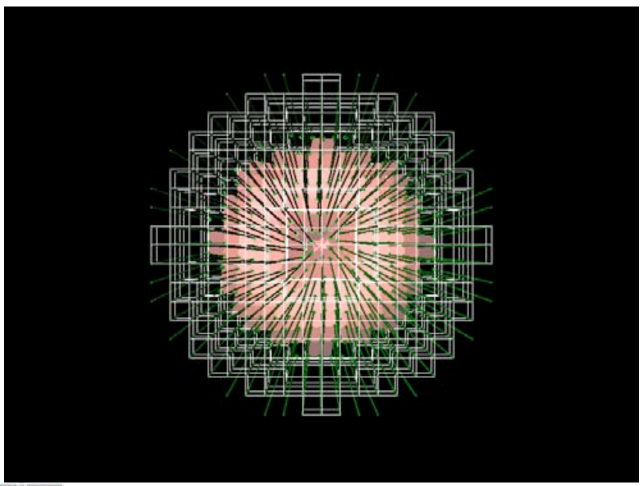
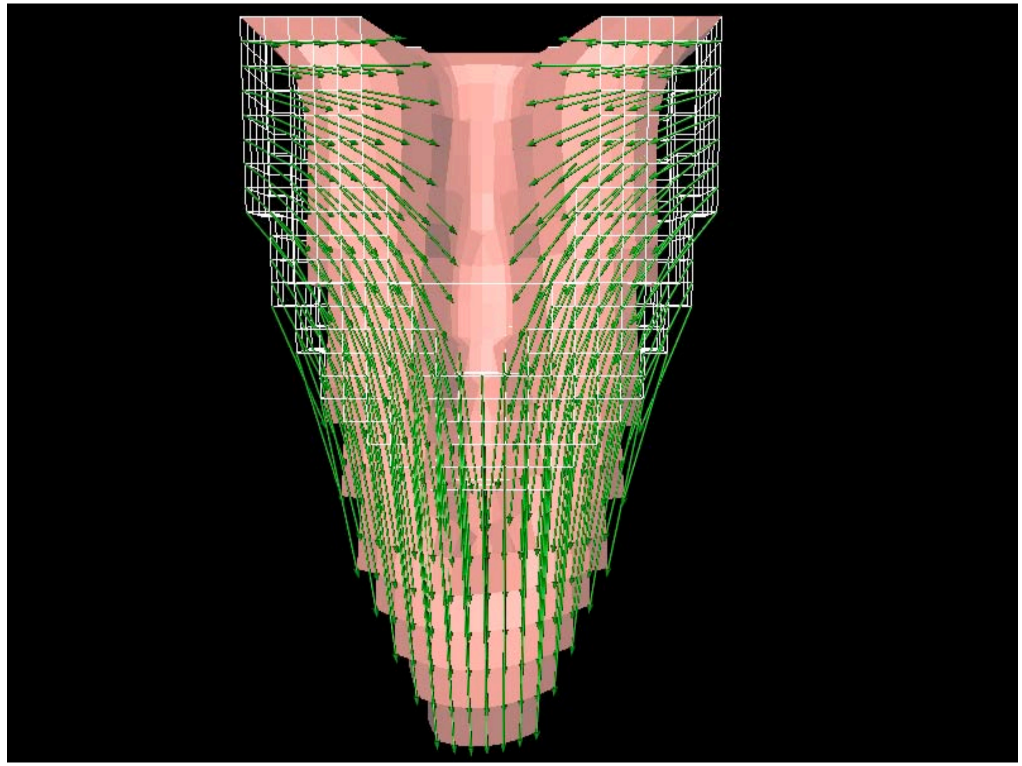
Example: 0°, 0°, 0°







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Group Work

Describe modules of a simulation software for cardiac electro-mechanics!

Which modules should be highly optimized?