Biomedical Optics V

Transport equation

Original slides: Stefan Andersson-Engels, Lund

IMT 2006-10-05

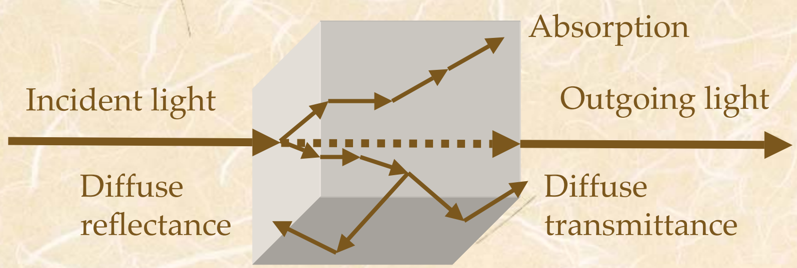
The Dualism of light

- *Light as waves:* light is considered as an electromagnetic wave modelled by the Maxwell equations
- *Light as particles:* light is considered as a stream of energetic particles - photons modelled by energy conservation - the transport equation

Questions

- Why can't you use lambert-Beer's law?
- Is it difficult to get images or physiological data from inside the tissue?

Light propagation in random media



Transport equation - examples

- heat conduction
- diffusion
- neutrons in nuclear reactions
- light propagation in turbid media

Transport equation

- Can be solved analytically
- Can be solved numerically

$\}$ Simplifications are necessary

- Can be used to simulate the transport

— Monte Carlo simulations

Monte Carlo simulations

- No simplifications necessary
- Provides results for a specific case only - no analytical expressions
- Photon statistics limit the signal-to-noise ratio
- Extensive computer capacity is often required

Radiance L(r,s,t)

- Radiance L(**r**,**s**,*t*) [W/m^{2}sr] is the quantity used to describe the propagation of photon power. It is defined as the radiant power per unit solid angle about the unit vector **s** and per unit area perpendicular to **s**.
- Radiance L(**r**,**s**,*t*) is related to the power dP(r,**s**,*t*) [W] flowing through infinitesimal area dA, located at **r**, in the direction of unit vector **s** (with an angle between **s** and the normal **n** of dA), within the solid angle Ω:

$$dP(\mathrm{r},\mathrm{s},t) = L(\mathrm{r},\mathrm{s},t)\,dA\cos\theta\, d\Omega$$

Radiant Energy Fluence Rate φ (r,t)

- An chromophore, located at **r**, can absorb photons irrespectively of their direction of propagation and the integral of the radiance over all directions, called the fluence rate φ(**r**) [W/m^2], has more practical significance than radiance itself.

$$\Phi(\mathrm{r},t) = \int_{4\pi} L(\mathrm{r},\mathrm{s},t)\,d\Omega = ch\nu \int_{4\pi} N(\mathrm{r},\mathrm{s},t)\,d\Omega$$

- The fluence rate is defined as the radiant power incident on a small sphere, divided by the cross sectional area of that sphere

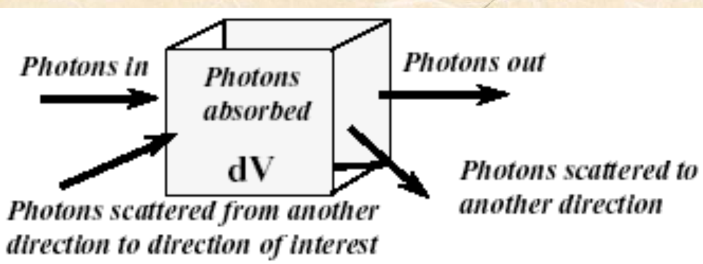
The photon distribution function N(r,s,t)

- N(r,s,t)d^{3}rdΩ is the number of photons in the volume d^{3}r with the direction **s** within dΩ at time t. The unit of N(**r**,**s**,*t*) is photons m^{-3}sr^{-1}. The radiance or intensity is obtained by multiplying N by the photon energy and the velocity of light:

$$L(\mathrm{r},\mathrm{s},t) = N(\mathrm{r},\mathrm{s},t)h\nu c$$

The transport equation (I)

- Assume a small volume dV and a direction s.
- Conservation of energy yields that photons can only be added to or subtracted from the photon distribution function in specific interactions.



Transport equation: 1:th term

- The time-resolved transport equation is a mathematical expression of the build-up of the photon density function N(**r**,**s**,*t*).
- Thus the first term expresses changes in the photon distribution function with time:

$$\int_V \frac{\partial N(\mathrm{r},\mathrm{s},t)}{\partial t}\,dV$$

Transport equation: 2:nd term

- Loss (I): Photons lost through the boundary. It can be expressed as a surface integral

$$-\oint_S cN(\mathrm{r},\mathrm{s},t)\,\mathrm{s}\bullet dS$$

- Using Gauss´ theorem we get

$$= -\int_V c\,\mathrm{s}\bullet\nabla N(\mathrm{r},\mathrm{s},t)dV$$

Transport equation: 3:d term

- Loss (II): Photons scattered from direction **s** to any another direction s′

$$= -\int_V c\mu_s(\mathrm{r})N(\mathrm{r},\mathrm{s},t)dV$$

Transport equation: 4:th term

- Loss (III): Photons incoming in direction **s** are absorbed

$$= -\int_V c\mu_a(\mathrm{r})N(\mathrm{r},\mathrm{s},t)dV$$

Transport equation: 5:th term

- Gain (I): Photons gained through scattering from any direction **s′** into the direction **s**

$$= +\int_V c\mu_s(\mathrm{r})\int_{4\pi} p(\mathrm{s}',\mathrm{s})N(\mathrm{r},\mathrm{s},t)ds'dV$$

Transport equation: 6:th term

- Gain (II): Photons gained through a light source q

$$+\int_V q(\mathrm{r},\mathrm{s},t)dV$$

The transport equation (II)

$$\int_V \frac{\partial N(\mathrm{r},\mathrm{s},t)}{\partial t}dV = -\int_V c\,\mathrm{s}\bullet\nabla N(\mathrm{r},\mathrm{s},t)dV$$
$$-\int_V c\mu_s(\mathrm{r})N(\mathrm{r},\mathrm{s},t)dV$$
$$-\int_V c\mu_a(\mathrm{r})N(\mathrm{r},\mathrm{s},t)dV$$
$$+\int_V c\mu_S(\mathrm{r})\int_{4\pi} p(\mathrm{s}',\mathrm{s})N(\mathrm{r},\mathrm{s},t)ds'dV$$
$$+\int_V q(\mathrm{r},\mathrm{s},t)dV$$

Energy conservation

- The transport equation is derived under the assumption that the energy of the non absorbed photons are kept the same despite all interactions with the medium, i.e. no loss or gain in photon energy due to the interaction.
- Second restriction is monochromatic light

Transport equation (III)

- The transport equation is usually presented for the radiance L(r,s,t) [W m^{-2} sr^{-1}], without the integrals

$$\frac{1}{c}\frac{\partial L(\mathrm{r},\mathrm{s},t)}{\partial t} + \mathrm{s}\cdot\nabla L(\mathrm{r},\mathrm{s},t) + (\mu_s + \mu_a)L(\mathrm{r},\mathrm{s},t) =$$
$$\mu_s\int_{4\pi} L(\mathrm{r},\mathrm{s}',t)p(\mathrm{s},\mathrm{s}')d\Omega + Q(\mathrm{r},\mathrm{s},t)$$

Transport equation (IIIb)

- The time-independent transport equation, valid for a steady-state situation, is derived by using that the radiance is independent of time, and thus is the time derivative of the radiance equal to zero

$$\mathrm{s} \cdot \nabla L(\mathrm{r},\mathrm{s}) = \frac{\partial L(r,s)}{\partial s} =$$

$$-(\mu_s + \mu_a)L(\mathrm{r},\mathrm{s}) + \mu_s \int_{4\pi} L(\mathrm{r},\mathrm{s}')\, p(\mathrm{s},\mathrm{s}')\, d\Omega + Q(\mathrm{r},\mathrm{s})$$

The total attenuation coefficient and albedo

- The total attenuation coefficient is defined as:

$$Albedo = \frac{\mu_s}{\mu_s + \mu_a}$$