Problems

4.1 In a BAEP investigation, the EPs are assumed to be modeled by (4.4) and related assumptions on statistical properties. The SNR of the first potential $\mathbf{x}_1$ is defined by

$$\text{SNR} = 10 \cdot \log \frac{\mathbf{s}^T \mathbf{s}}{E[\mathbf{v}_1^T \mathbf{v}_1]}$$

and is assumed to be equal to –5 dB. All other EPs in the ensemble have identical SNRs. How many EPs need to be averaged, using (4.12), in order to increase the SNR to 10 dB?

4.2 The difference between two subaverages $\hat{s}_{a_0}(n)$ and $\hat{s}_{a_1}(n)$ is denoted

$$\Delta \hat{s}_a(n) = \hat{s}_{a_0}(n) - \hat{s}_{a_1}(n).$$

The two subaverages have been obtained by splitting the ensemble in a suitable way.

a. Explain why it is of interest to study the quantity $\Delta \hat{s}_a(n)$ during the acquisition of EPs.

b. Show that the variance of $\Delta \hat{s}_a(n)$ is equal to $4\sigma_v^2/M$ by making use of the common assumptions associated with ensemble averaging.

4.3 Determine the impulse response of the exponential averager in (4.35). The answer should be expressed as a function of the weight factor α and the length N of the EP. Assume that $\hat{s}_{e,0}(n) = 0$, and recall that all EPs are concatenated. Sketch the impulse response.

4.4 Computation of the ensemble variance estimate $\hat{\sigma}_v^2(n)$ in (4.17) has the disadvantage of requiring that the entire ensemble must be available before $\hat{\sigma}_v^2(n)$ can be computed. However, it may by desirable to monitor how the ensemble variance evolves as the number of EPs increases. Derive an approximate estimator which recursively computes the estimate of $\hat{\sigma}_{v,M}^2(n)$. It can be assumed that the ensemble average $\hat{s}_{a,M}(n)$ has been stabilized to such a degree that it can be approximated by its preceding estimate $\hat{s}_{a,M-1}(n)$.

4.5 Determine the mean of the exponential averager $\hat{s}_{e,M}(n)$ when $\hat{s}_{e,0}(n) = x_1(n)$. Discuss the fact that $E[\hat{s}_{e,M}(n)]$ is unbiased, whereas it is asymptotically unbiased when $\hat{s}_{e,0}(n) = 0$.

4.6 a. The exponential averager is usually initialized by either $\hat{s}_{e,0}(n) = 0$ or $\hat{s}_{e,0}(n) = x_1(n)$. However, both these initializations suffer from certain disadvantages. What are these disadvantages?

b. Find that value of α of the exponential averager which makes the variance of $\hat{s}_{e,M}(n)$ equal to the variance of the ensemble averager $\hat{s}_{a,M}(n)$.

4.7 Derive the expression of the variance $V[\hat{s}_{e,M}(n)]$ in (4.38).

4.8 Determine the width of the frequency lobes of the comb filter at the –3 dB point corresponding to:

a. the ensemble averager as a function of M and N, and

b. the exponential averager as a function of α.

In both these cases, it is assumed that the poles are well-separated such that the influence of neighboring poles can be neglected. Compare the role of M and α of the respective estimators.

4.9 Determine a closed-form expression for the –3 dB bandwidth of the peaks in the exponential averager, expressed in terms of the parameters α and N.

4.10 In addition to determining the magnitude function of the ensemble averager, cf. page 202, it is also of interest to determine its phase function.

a. Derive an expression for the phase function of the ensemble averager and plot it.

b. Discuss how the phase function influences the repetitive signal and the noise, respectively.

4.11 An anesthetized patient is periodically stimulated by short sound pulses to continuously monitor the BAEP. The resulting waveforms first stabilize at an amplitude of 0.6 μV in peak IV, but then suddenly decrease to an amplitude of 0.2 μV.

a. For ensemble averaging, determine the delay in terms of the number of stimuli until the amplitude (in the mean) has dropped below 0.3 μV?

b. Repeat the exercise in (a) for exponential averaging.

4.12 The ensemble average $\hat{s}_{a,M}(n)$ is often used to estimate the signal $s(n)$ in the observation model $x_i(n) = s(n) + v_i(n)$, where $v_i(n)$ is zero-mean noise with variance σ_v^2. The ensemble average can be computed recursively using the following expression

$$\hat{s}_{a,M}(n) = \hat{s}_{a,M-1}(n) + g_M(x_M(n) - \hat{s}_{a,M-1}(n)),$$

where $g_M = 1/M$. Analogously, the weighted average $\hat{s}_{w,M}(n)$ can be computed recursively but using another expression of g_M. Determine g_M for weighted averaging under the assumption that the noise variance is $\sigma^2_{v_i}$, and then determine the recursion for $\sigma^2_{v_i} \equiv \sigma^2_v$.

4.13 In the interval preceding the stimulus elicited at time $n = 0$, we want to estimate the variance σ^2_v from the background EEG signal, e.g., for later use in the computation of the weighted average. It is assumed that the samples $x(-N), \ldots, x(-1)$, are modeled as uncorrelated, Gaussian noise with mean m_v and variance σ^2_v. Determine the ML estimator of σ^2_v.

4.14 Determine the expression with which the variance of the weighted average $V[\hat{s}_{w,M}(n)]$ can be recursively computed from $V[\hat{s}_{w,M-1}(n)]$. The variance $V[\hat{s}_{w,M}(n)]$ is given in (4.68).

4.15 Determine $E[\hat{s}_w(n)]$ and $V[\hat{s}_w(n)]$ for weighted averaging under the assumption that the signal amplitude varies and the noise variance remains fixed for all EPs. Comment on bias and consistency.

4.16 Derive the optimal weights $\hat{w}_i$ of the weighted average that minimizes the following MSE criterion,

$$E\left[\left(s(n) - \sum_{i=1}^{M} w_i x_i(n)\right)^2\right].$$

Each EP is described by $x_i(n) = s(n) + v_i(n)$, where $s(n)$ is deterministic and $v_i(n)$ is random with variance $\sigma^2_{v_i}$.

a. Determine the optimal weights, and comment on their dependence on the signal and noise.

b. Show that the optimal weights approach those in (4.67) when the constraint

$$\sum_{i=1}^{M} w_i = 1$$

is introduced; this constraint assures that the ensemble average is unbiased.

4.17 Two cases of weighted averaging have been described in the text—either varying signal amplitude or varying noise variance. In this problem, a third case is examined where both amplitude and noise variance are allowed to vary. Find the optimal weight vector for this case.

4.18 Weighted averaging requires that the noise variance of each EP be estimated. Although the estimator in (4.73) is adequate for certain applications such as BAEP and SEP, it is less suitable for VEPs where the SNR is relatively good. Suggest a variance estimator for the latter case which draws upon the better SNR.

4.19 The weights required in weighted averaging can be adaptively estimated by taking advantage of the assumption that signal and noise are uncorrelated [53]. The estimation is based on the adaptive linear combiner, shown in Figure 3.13, but now with the primary input (i.e., the upper branch of the block diagram) given by the ensemble average $\hat{s}_a(n)$ and the M reference inputs given by $x_i(n) = s(n) + v_i(n)$ for $i = 1, \ldots, M$.

a. Assuming a steady-state situation, show that the LMS algorithm converges in the mean to the optimum weight vector $\mathbf{w}^{\mathrm{o}}(n)$, cf. (3.55),

$$\mathbf{w}^{\mathrm{o}}(n) = \frac{s^2(n)}{1 + \displaystyle\sum_{i=1}^{M} \frac{s^2(n)}{\sigma_i^2}} \begin{bmatrix} \frac{1}{\sigma_1^2} & \frac{1}{\sigma_2^2} & \cdots & \frac{1}{\sigma_M^2} \end{bmatrix}^T .$$

b. Unfortunately, the weight vector that results in (a) is time-varying through $s(n)$ despite the fact that the noise is assumed to be stationary. As a result, the weight vector obtained by the LMS algorithm is biased. By introducing the constraint

$$\mathbf{w}^T\mathbf{1} = 1, \tag{4.376}$$

which assures that the estimate is unbiased, a constrained LMS algorithm can be developed which minimizes the MSE,

$$\mathcal{E}_{\mathbf{w}} = E\left[\left(\hat{s}_a(n) - \mathbf{w}^T\mathbf{x}(n)\right)^2\right] - \lambda(\mathbf{w}^T\mathbf{1} - 1),$$

where the constraint is multiplied by the Lagrange multiplier λ. Derive the constrained LMS algorithm.

4.20 Another estimate of the normalization constant in (4.78) is given by

$$\widehat{\mathbf{a}^T\mathbf{a}} = \mathrm{tr}(\mathbf{X}^T\mathbf{X}).$$

Explain why this estimate is less suitable than the one given in the text.

4.21 The ML estimator of a signal corrupted by stationary (i.e., $\sigma_{v_i}^2 \equiv \sigma_v^2$), Laplacian noise is the ensemble median. Determine the ML estimator—the *weighted median*—when the Laplacian noise has a variance which varies from potential to potential.

4.22 Show that the recursive, robust averager with outlier rejection in (4.104), whose influence function is given by the sgn function, tends to converge to the median.

4.23 Determine an approximate expression for the –3 dB cut-off frequency F_c of the lowpass filter in Figure 4.20(b), assuming that the latency shifts τ are uniformly distributed. In other words, determine that Ω_c $(= 2\pi F_c)$ for which

$$P_\tau(\Omega_c) = \frac{\sin \frac{1}{2}\Omega_c T}{\frac{1}{2}\Omega_c T} = \frac{1}{\sqrt{2}}.$$

4.24 For discrete-time jitter, show that the characteristic function $P_\theta(e^{\jmath\omega})$ of the "discretized" Gaussian PDF,

$$p_\theta(\theta) = \frac{1}{\sqrt{2\pi\sigma_\theta^2}}\, e^{-\frac{\theta^2}{2\sigma_\theta^2}},$$

where θ is an integer, is given by

$$P_\theta(e^{\jmath\omega}) = \sum_{n=-\infty}^{\infty} e^{-\sigma_\theta^2(\omega-2\pi n)^2/2}.$$

For Gaussian jitter, the resulting effect is lowpass filtering with a cut-off frequency ω_c of

$$\omega_c = \frac{0.83}{\sigma_\theta}.$$

4.25 Latency estimation based on cross-correlation of $x(n)$ of length N and a deterministic waveform $s(n)$ of length $M < N$ may be formulated as

$$\hat{\tau} = \arg \max_{n_0 \in [0, N-M]} \sum_{n=n_0}^{n_0+M-1} x(n)s(n-n_0),$$

where $\hat{\tau}$ is the estimated latency, i.e., the argument which maximizes the above cross-correlation. Suggest two different techniques for latency estimation with better time resolution than that offered by the sampling interval of the original signal.

4.26 Derive a filter $h(n)$ of length N that maximizes the SNR at $n = N - 1$ for the model

$$x(n) = s(n) + v(n), \quad n = 0, 1, \ldots, N-1,$$

where $s(n)$ is deterministic and $v(n)$ is stationary, colored, zero-mean, Gaussian noise with correlation matrix $\mathbf{R}_v$.

4.27 Derive the ML estimator of the delay θ_i when the EP is corrupted by stationary, colored, zero-mean, Gaussian noise with correlation matrix $\mathbf{R}_v$. Assuming that the signal length N is much larger than the correlation time d for the noise $v(n)$ and $r_v(k) = 0$ for $|k| \geq d$ (the so-called *asymptotic Gaussian PDF* assumption), it can be shown that the inverse of the noise correlation matrix $\mathbf{R}_v^{-1}$ is given by [181, p. 33]

$$\mathbf{R}_v^{-1} = \sum_{i=0}^{N-1} \frac{1}{\lambda_i} \boldsymbol{\varphi}_i \boldsymbol{\varphi}_i^T,$$

where λ_i is an eigenvalue of $\mathbf{R}_v$ for which the corresponding eigenvectors are given by the discrete Fourier transform vector,

$$\boldsymbol{\varphi}_i = \frac{1}{\sqrt{N}} [\begin{array}{ccccc} 1 & e^{j2\pi f_i} & e^{j4\pi f_i} & \cdots & e^{j2\pi(N-1)f_i} \end{array}]^T.$$

Make use of this result in the derivation of the ML estimator.

4.28 A multichannel variant of the Woody method may be used in which each channel $x_i(l)$ is first processed by its corresponding matched filter $h_i(l)$, followed by a weighted summation of the filter outputs which is used for time delay estimation, i.e.,

$$y(n) = \sum_{i=1}^{P} \beta_i \left(\sum_{l=0}^{K} h_i(l) x_i(n-l) \right),$$

where P denotes the number of channels. Discuss, in general terms, how to choose the channel weights β_i.

4.29 The inverse of the correlation matrix $\mathbf{R}_x$ in (4.149) is required for weighting of the samples in the averaged EPs with the ML estimate of the ensemble correlation. The correlation matrix $\mathbf{R}_x$ can be expressed as

$$\mathbf{R}_x = (1 - \rho(n))\mathbf{I} + \rho(n)\mathbf{1}\mathbf{1}^T,$$

where, for simplicity, it is assumed that the power of the observed signal is normalized to unity, i.e., $\sigma_s^2(n) + \sigma_v^2 = 1$. Find the inverse $\mathbf{R}_x^{-1}$ expressed in terms of $\rho(n)$ and the number M of EPs. *Hint:* Use the matrix inversion lemma in (A.31).

4.30 In weighting an averaged EP with the ensemble correlation, we have assumed that $s(n)$ is random. An alternative approach is to assume that $s(n)$ is deterministic (once the ensemble has been fixed).

a. Show that the weight minimizing the MSE criterion is given by

$$w(n) = \frac{s^2(n)}{s^2(n) + \frac{\sigma_v^2}{M}}.$$

b. Propose estimators of $s(n)$ and σ_v^2 in order to determine the weight $w(n)$ in (a).

4.31 When performing single-trial analysis, it may be of interest to minimize the following MSE criterion:

$$\mathcal{E}_{\mathbf{w}} = E\left[||\mathbf{x}_i - \mathbf{\Phi}\mathbf{w}_i||^2\right].$$

In the text, it was tacitly assumed that the obtained solution in (4.202) corresponded to the minimum of the MSE. Show that this solution really corresponds to the minimum.

4.32 An estimate of the signal correlation matrix $\mathbf{R}_s$ is required for implementation of the a posteriori FIR Wiener filter in (4.184). One approach to develop an estimator is based on the model $\mathbf{x}_i = \mathbf{s} + \mathbf{v}_i$, where it is assumed that $\mathbf{s}$ is stationary and $\mathbf{v}_i$ is uncorrelated from EP to EP.

a. Suggest an estimator which involves the summation of all cross-products $\mathbf{x}_i\mathbf{x}_j^T$ for $i, j = 1, \ldots, M$, while excluding $i = j$.

b. For this estimator, evaluate its behavior in terms of mean and variance.

4.33 Rather than minimizing the squared error over all possible realizations, as done in (4.199), we can minimize the "instantaneous" error

$$\mathcal{E}(\mathbf{w}_i) = \|\mathbf{x}_i - \mathbf{\Phi}\mathbf{w}_i\|^2$$

for one particular realization of $\mathbf{x}_i$. Proceeding in a way similar to the minimization of (4.199), the solution to this problem is found to be

$$\hat{\mathbf{w}}_i = \mathbf{\Phi}^T\mathbf{x}_i,$$

which is identical to the right-hand side of (4.203).