EVOKED POTENTIALS

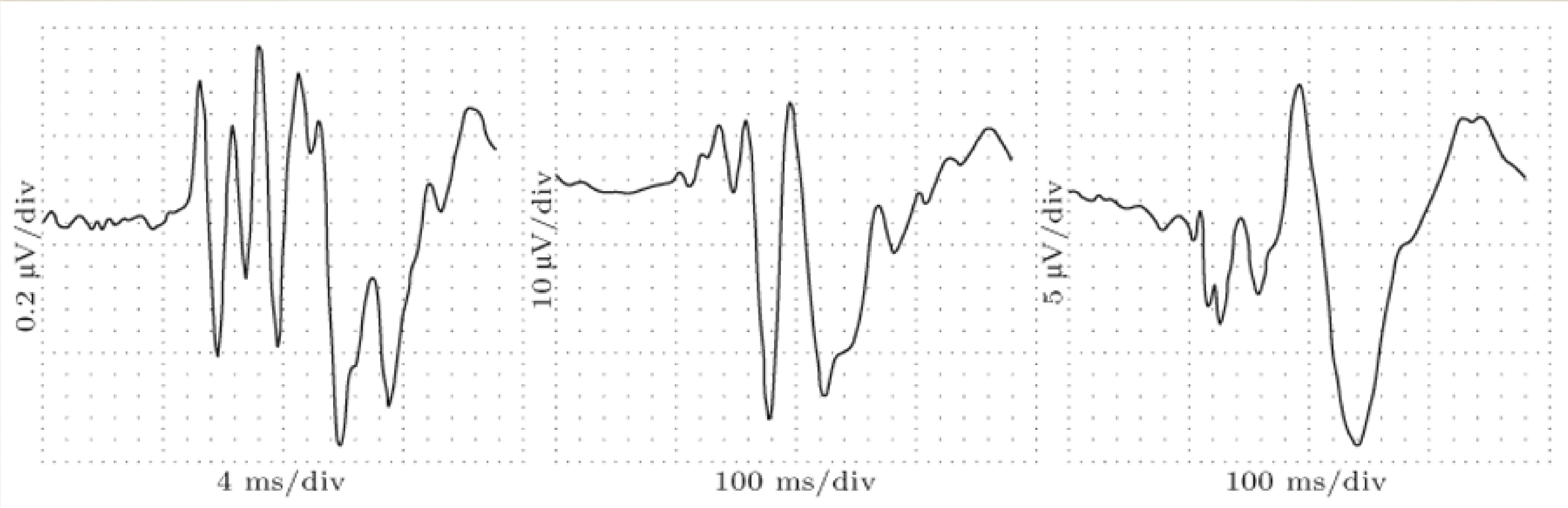
Evoked Potentials (EPs)

* Event-related brain activity where the stimulus is usually of sensory origin.
* Acquired with conventional EEG electrodes.
* Time-synchronized = time interval from stimulus to response is usually constant.

EP = A Transient Waveform

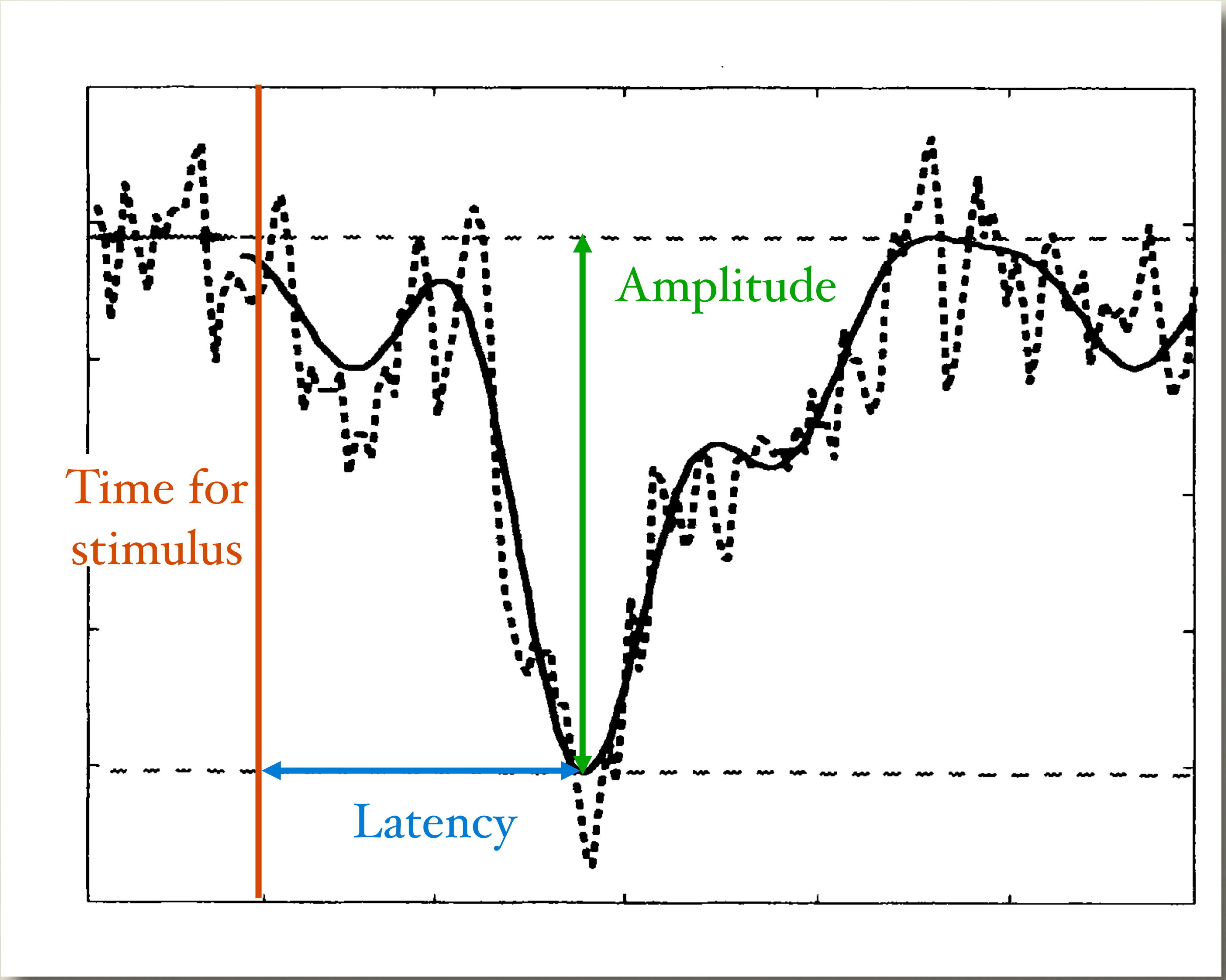
- Evoked potentials are usually ”hidden" in the EEG signal.
- Their amplitude ranges from 0.1–10 μV, to be compared with 10–100 μV of the EEG.
- Their duration is 25–500 milliseconds.

Examples of Evoked Potentials

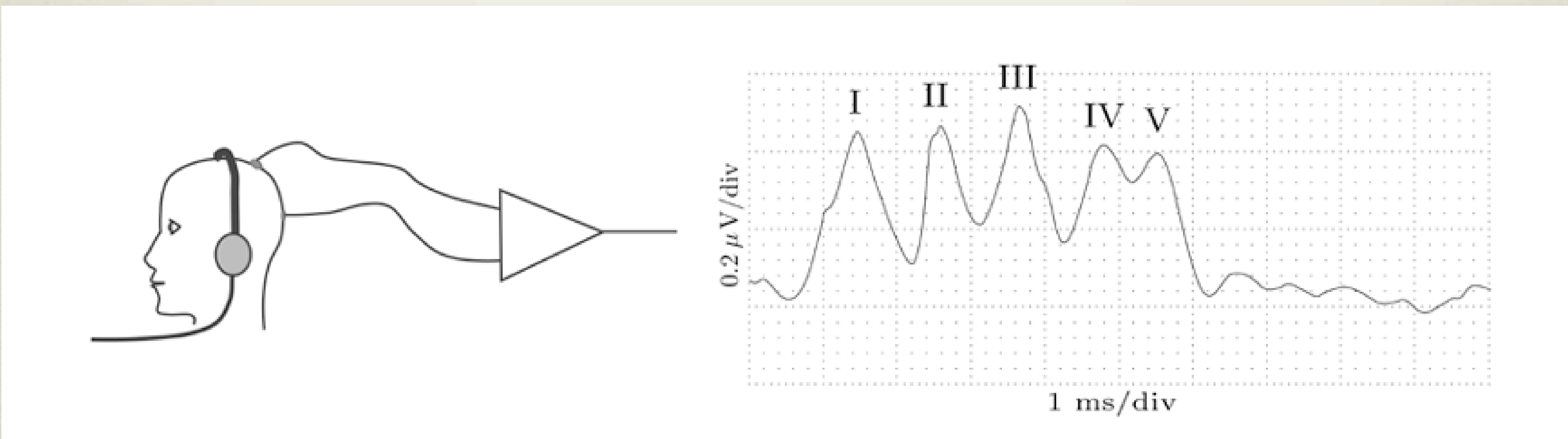


Note the widely different amplitudes and time scales.

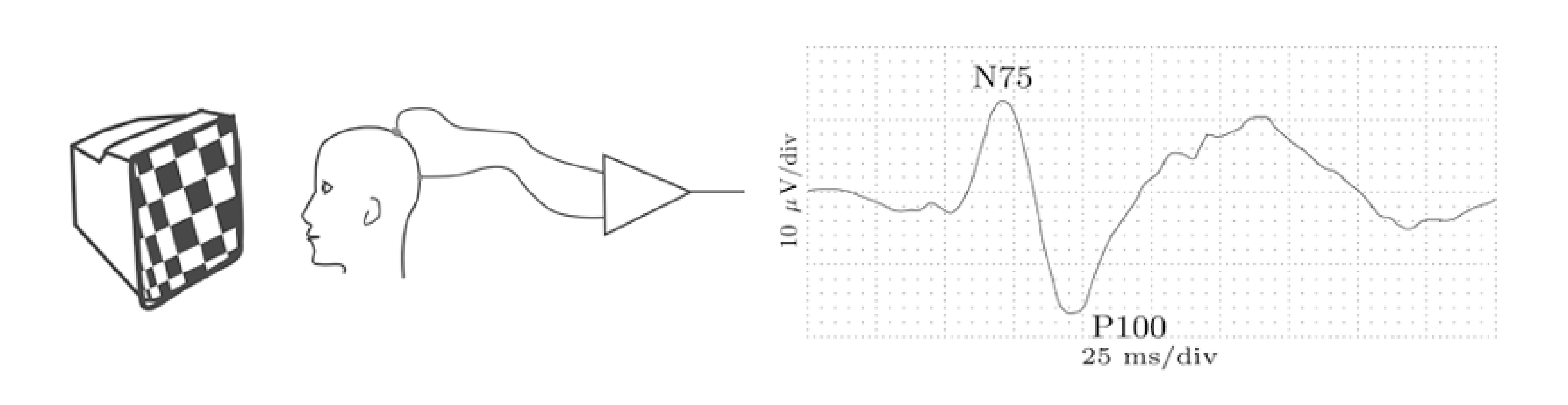
EP – Definitions



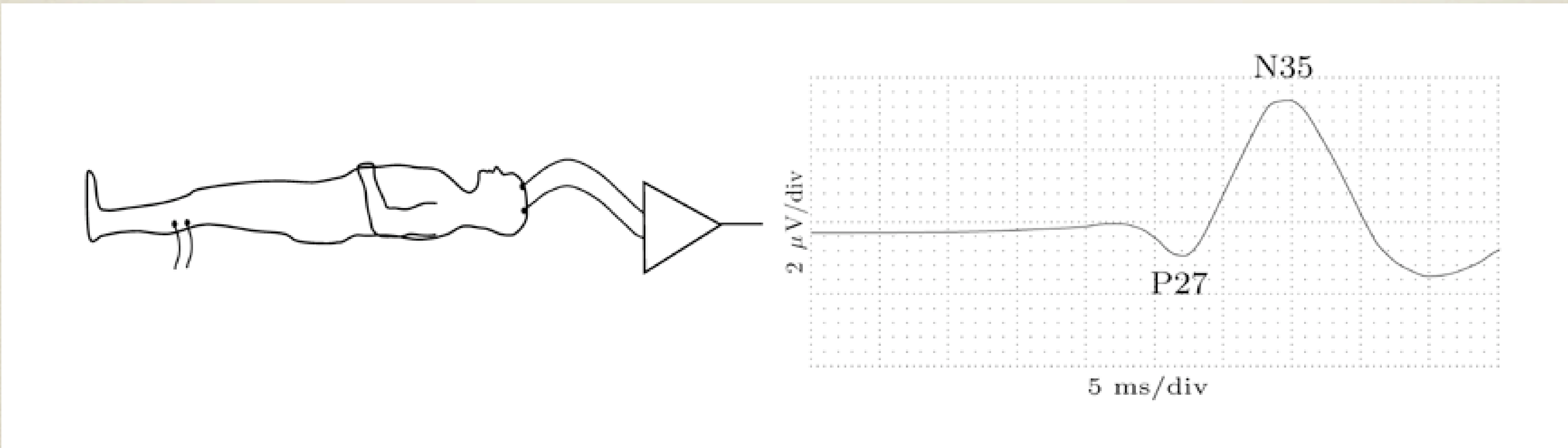
Auditive Evoked Potentials– AEPs



Visual Evoked Potentials–VEPs

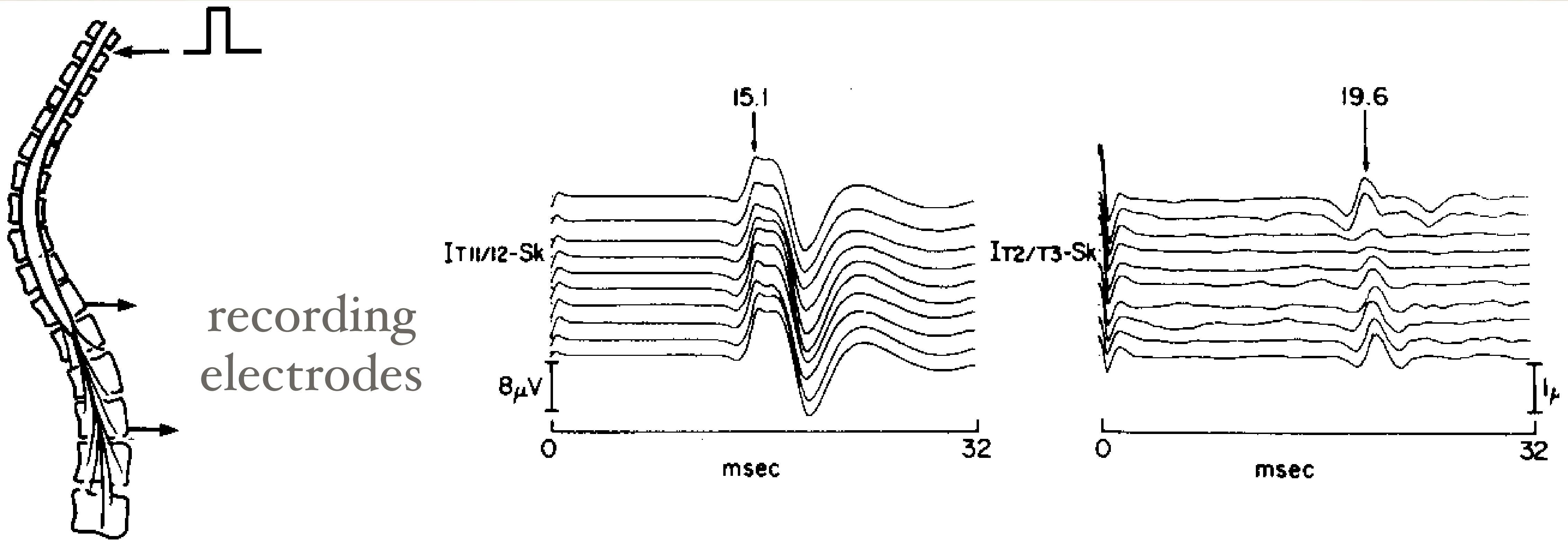


Somatosensory Evoked Potentials–SEPs



SEPs during Spinal Surgery

stimulation

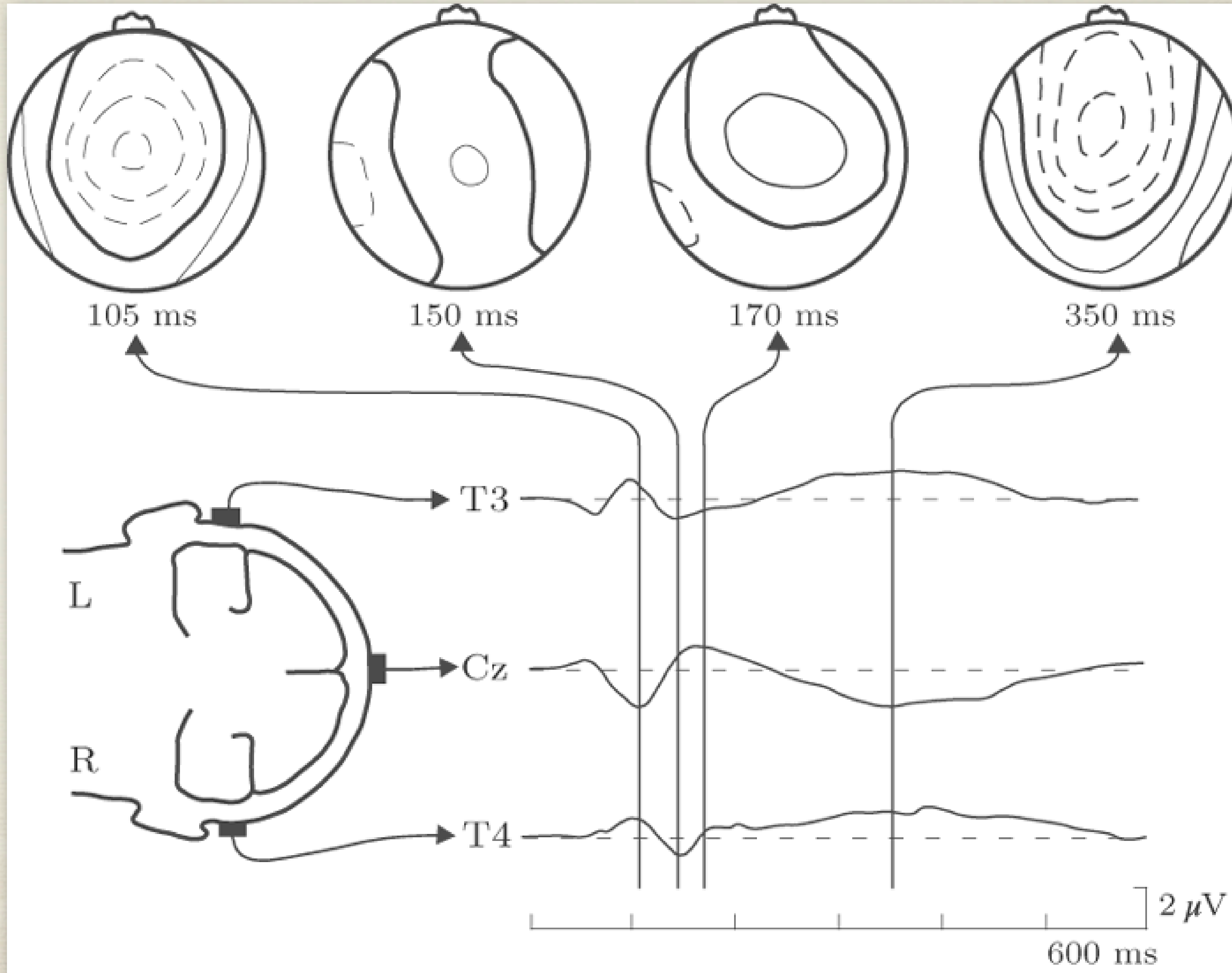


recording electrodes

electrode #1

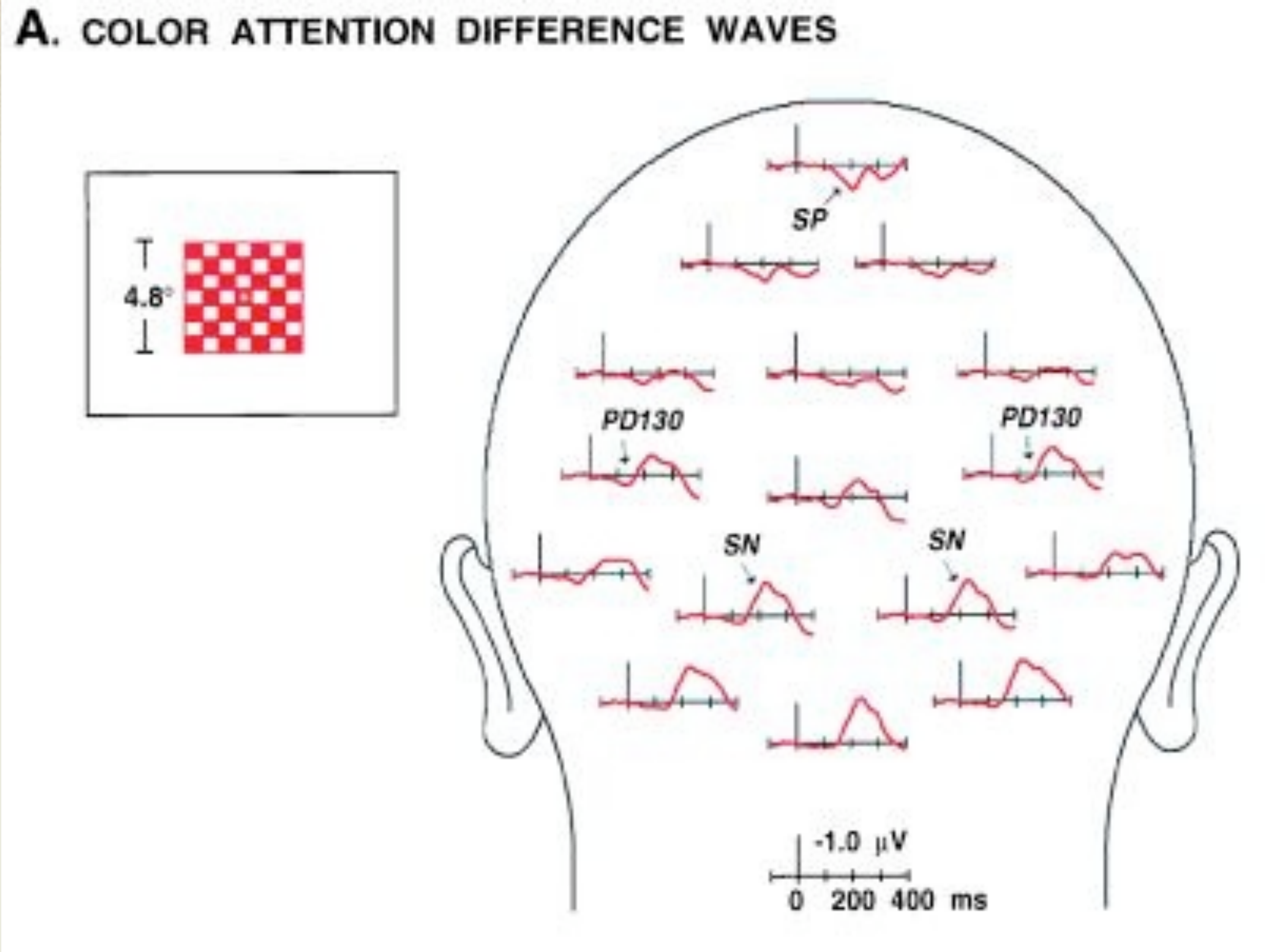
electrode #2

EP Scalp Distribution

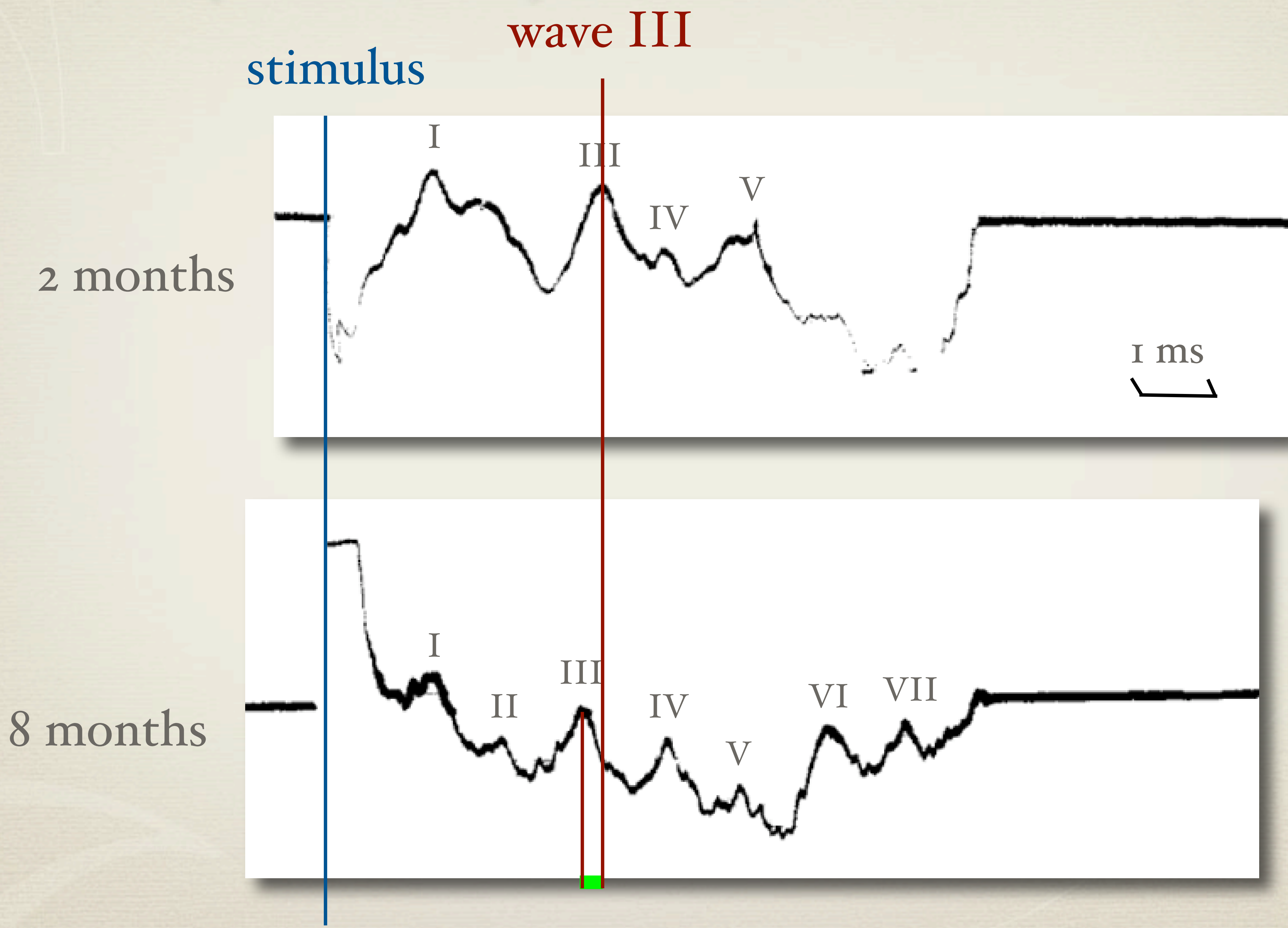


A. Evoked potentials resulting from a color task in which red and blue flashed checkerboards were presented in a rapid, randomized sequence at the center of the screen.

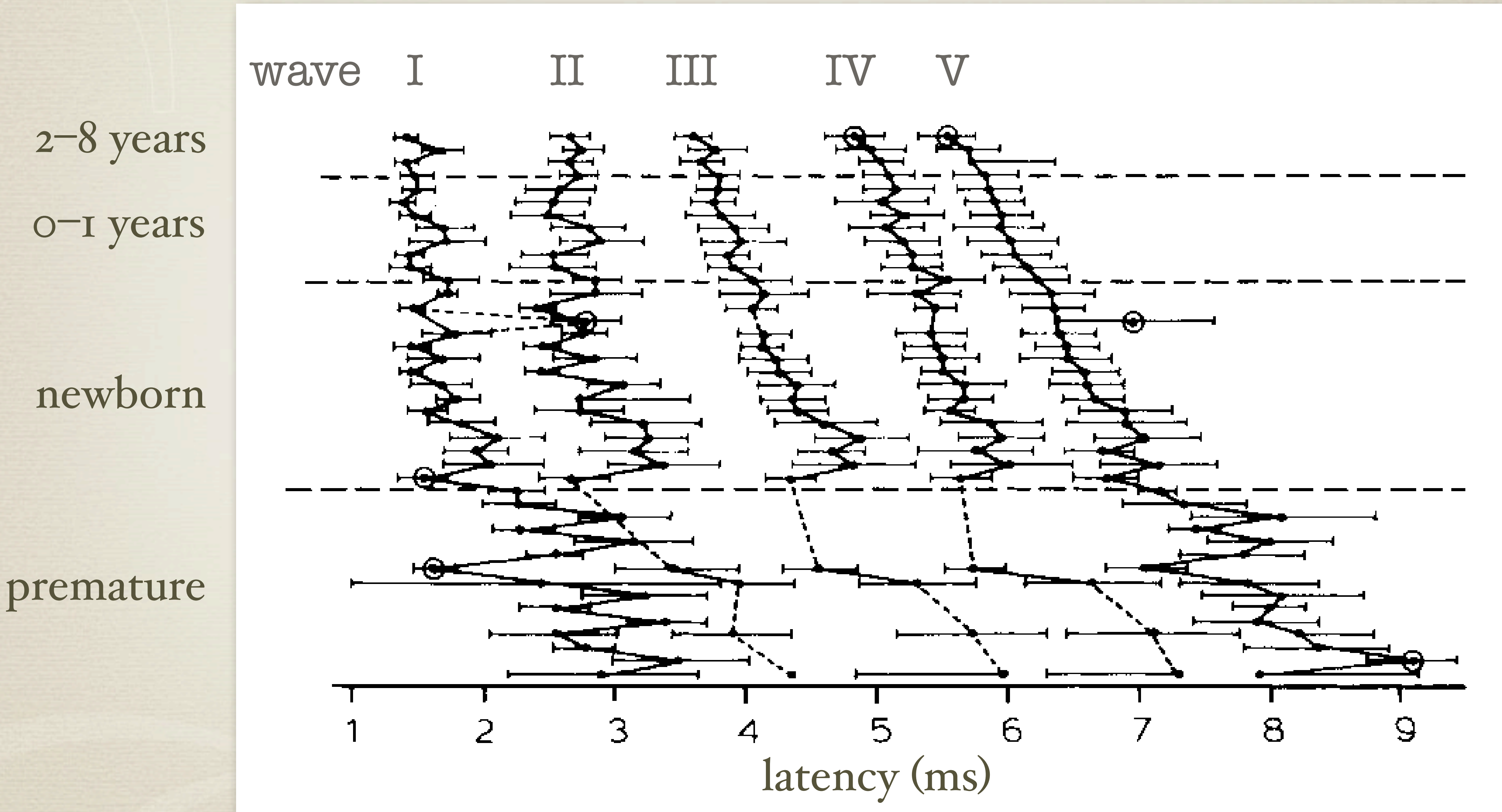
B. Scalp voltage distributions evoked potentials at different latency ranges.



Brainstem Auditive EP (BAEPs) in Newborns



BAEPs of Healthy Children



Cognitive EPs

'They wanted to make the hotel look like a tropical resort. So along the driveway they planted rows of ....'

Ensemble Formation

The observed EEG signal can be transformed into an ensemble of M different potentials, with each potential $x_i(n)$ described by N samples,

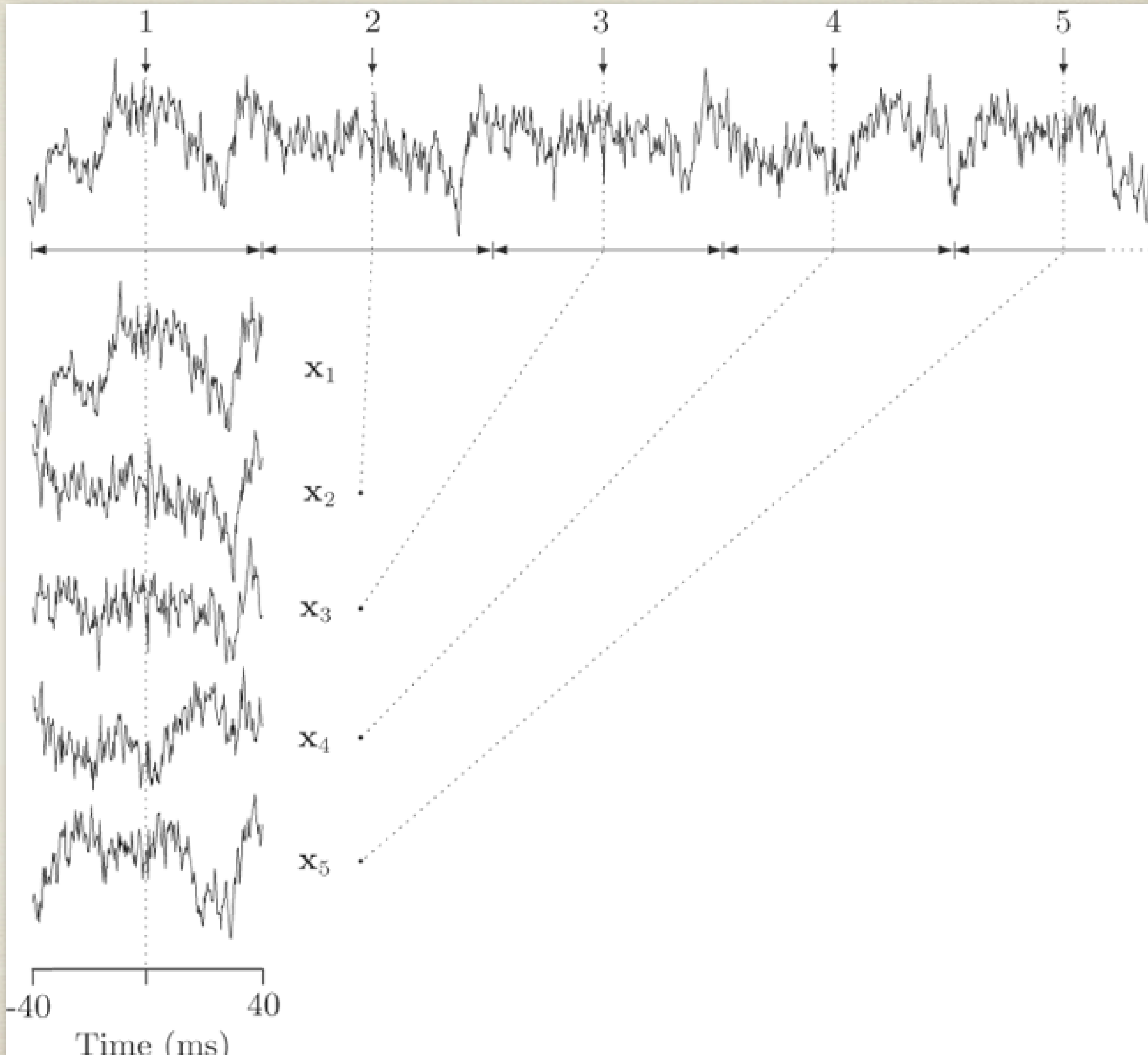
$$x_i(n), \quad i = 1, \ldots, M; \; n = 0, \ldots, N-1. \tag{4.1}$$

Formation of an EP Ensemble

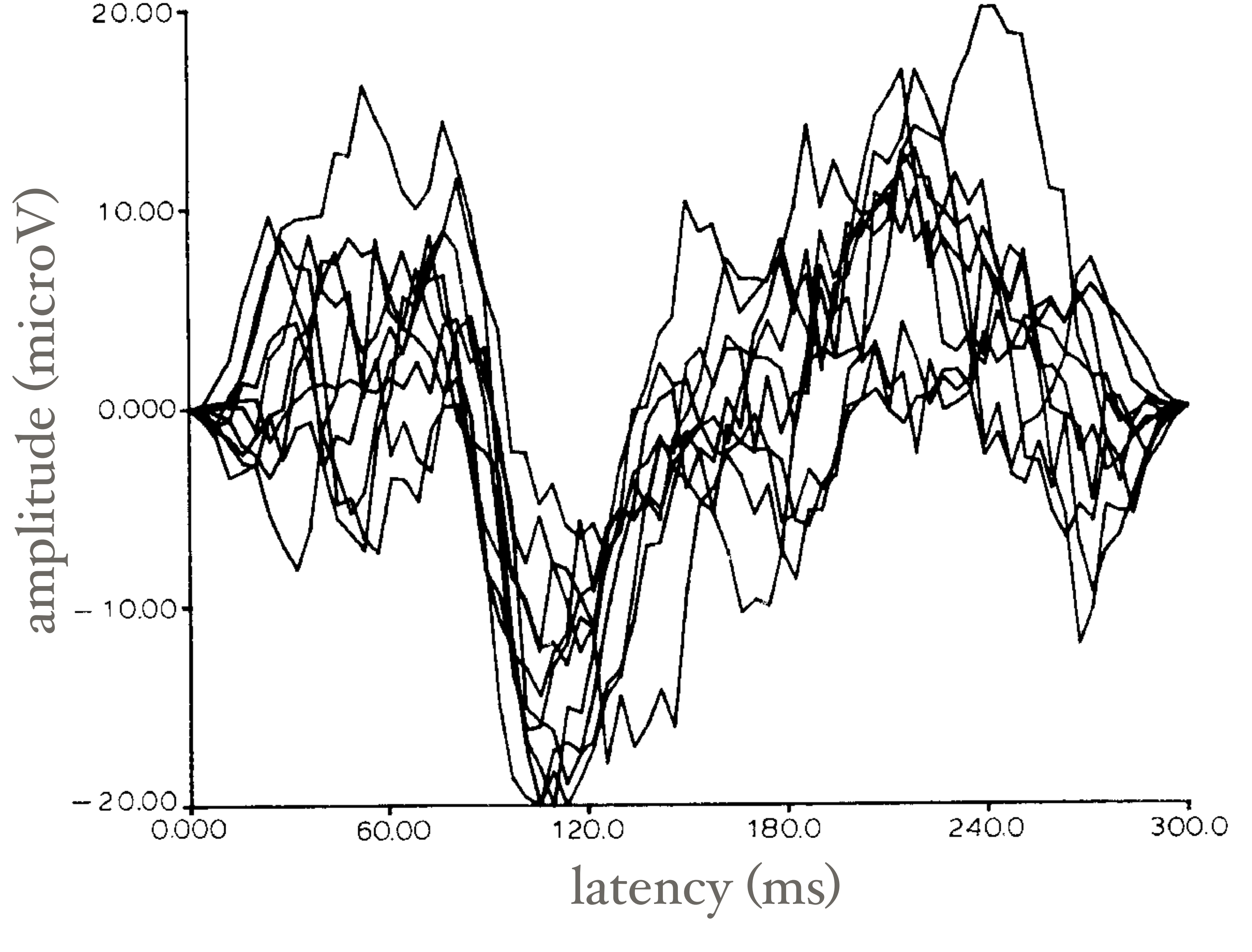
stimulus#

EEG signal

EP ensemble



10 Superimposed EPs



Model for Ensemble Averaging

Ensemble averaging is based on a simple signal model in which the potential $\mathbf{x}_i$ of the i^{th} stimulus is assumed to be additively composed of a deterministic, evoked signal component $\mathbf{s}$ and random noise $\mathbf{v}_i$ which is asynchronous to the stimulus,

$$\mathbf{x}_i = \mathbf{s} + \mathbf{v}_i, \tag{4.4}$$

fixed shape

where

$$\mathbf{s} = \begin{bmatrix} s(0) \\ s(1) \\ \vdots \\ s(N-1) \end{bmatrix}. \tag{4.5}$$

Noise Assumptions

I.

II.

III.

The noise $\mathbf{v}_i$ of the i^{th} EP,

$$\mathbf{v}_i = \begin{bmatrix} v_i(0) \\ v_i(1) \\ \vdots \\ v_i(N-1) \end{bmatrix}, \tag{4.6}$$

is assumed to derive from the ongoing "noise" process $v(n)$ which, in this model, is a stationary, zero-mean process,

$$E[v(n)] = 0. \tag{4.7}$$

Ensemble Averaging

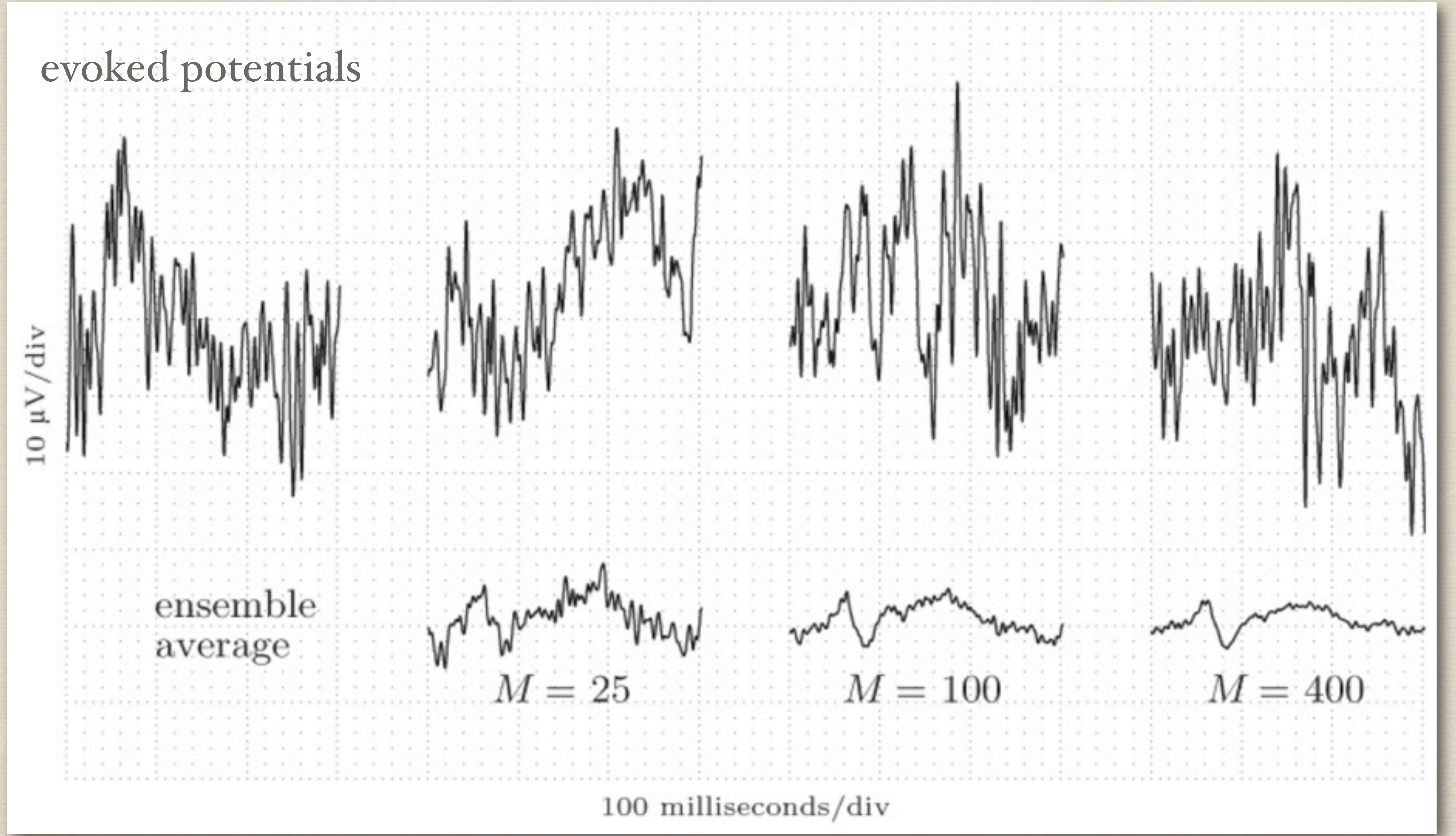
The ensemble average is defined by

$$\hat{\mathbf{s}}_a = \frac{1}{M}(\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_M) = \frac{1}{M}\mathbf{X}\mathbf{1} \qquad (4.12)$$

The more familiar (scalar) expression for ensemble averaging is given by

$$\hat{s}_a(n) = \frac{1}{M}\sum_{i=1}^{M} x_i(n),$$

Ensemble Averaging



Noise Variance

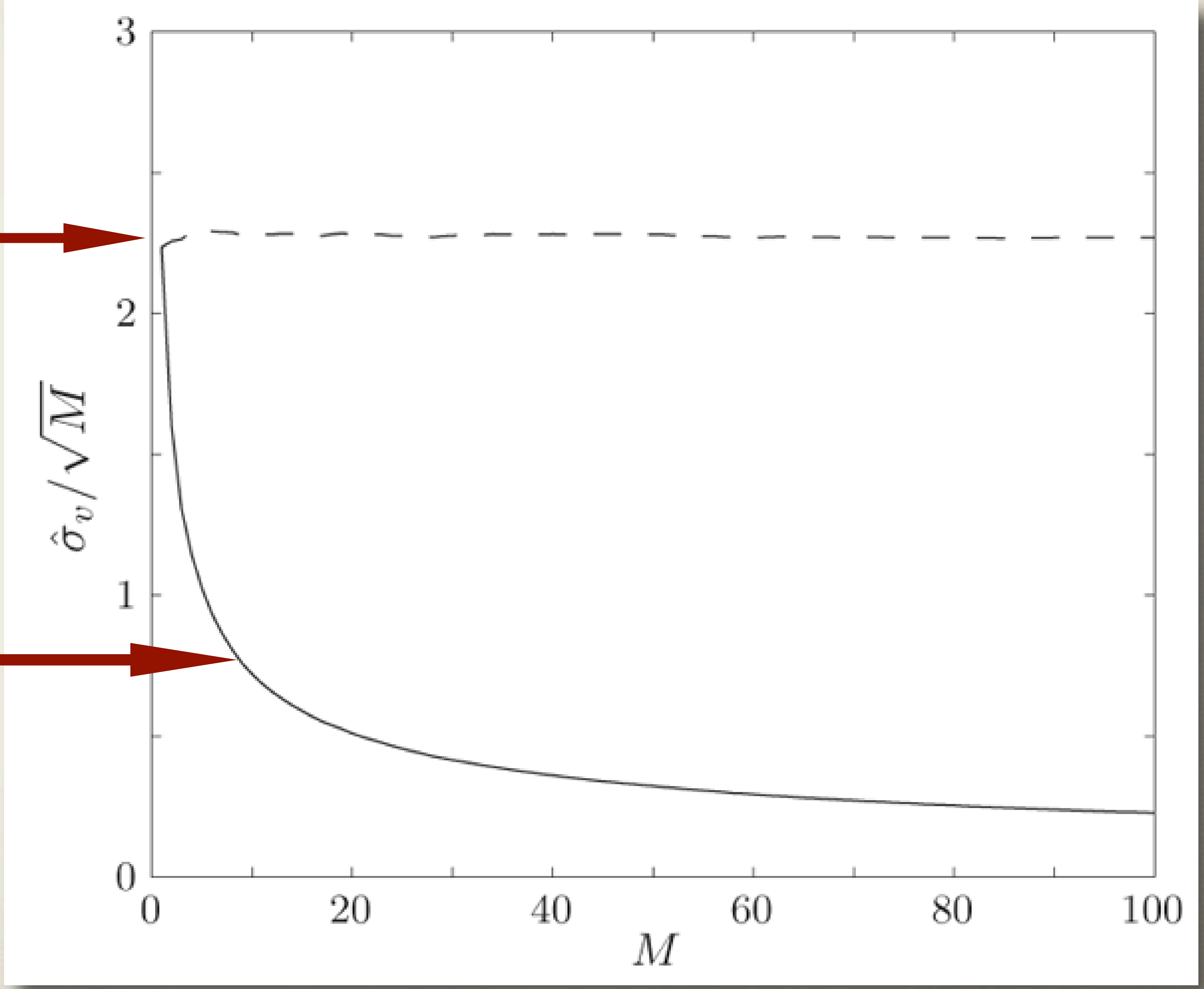
The variance of the ensemble average is inversely proportional to the the number of averaged potentials, that is:

$$\begin{aligned}
V[\hat{s}_a(n)] &= E\left[(\hat{s}_a(n) - E[\hat{s}_a(n)])^2\right] \\
&= \frac{1}{M^2}\sum_{i=1}^{M}\sum_{j=1}^{M} E[v_i(n)v_j(n)] \\
&= \frac{\sigma_v^2}{M}.
\end{aligned}$$

Reduction of Noise Level

The noise estimate before division by the reduction factor $1/\sqrt{M}$

Reduction in noise level of the ensemble average as a function of #potentials



#potentials

Exponential averaging

The ensemble average can be computed recursively because:

$$
\begin{aligned}
\hat{\mathbf{s}}_{a,M} &= \frac{1}{M}\mathbf{X}_M \mathbf{1}_M \\
&= \frac{1}{M}(\mathbf{X}_{M-1}\mathbf{1}_{M-1} + \mathbf{x}_M) \\
&= \hat{\mathbf{s}}_{a,M-1} + \frac{1}{M}(\mathbf{x}_M - \hat{\mathbf{s}}_{a,M-1}), \quad M \geq 1.
\end{aligned}
$$

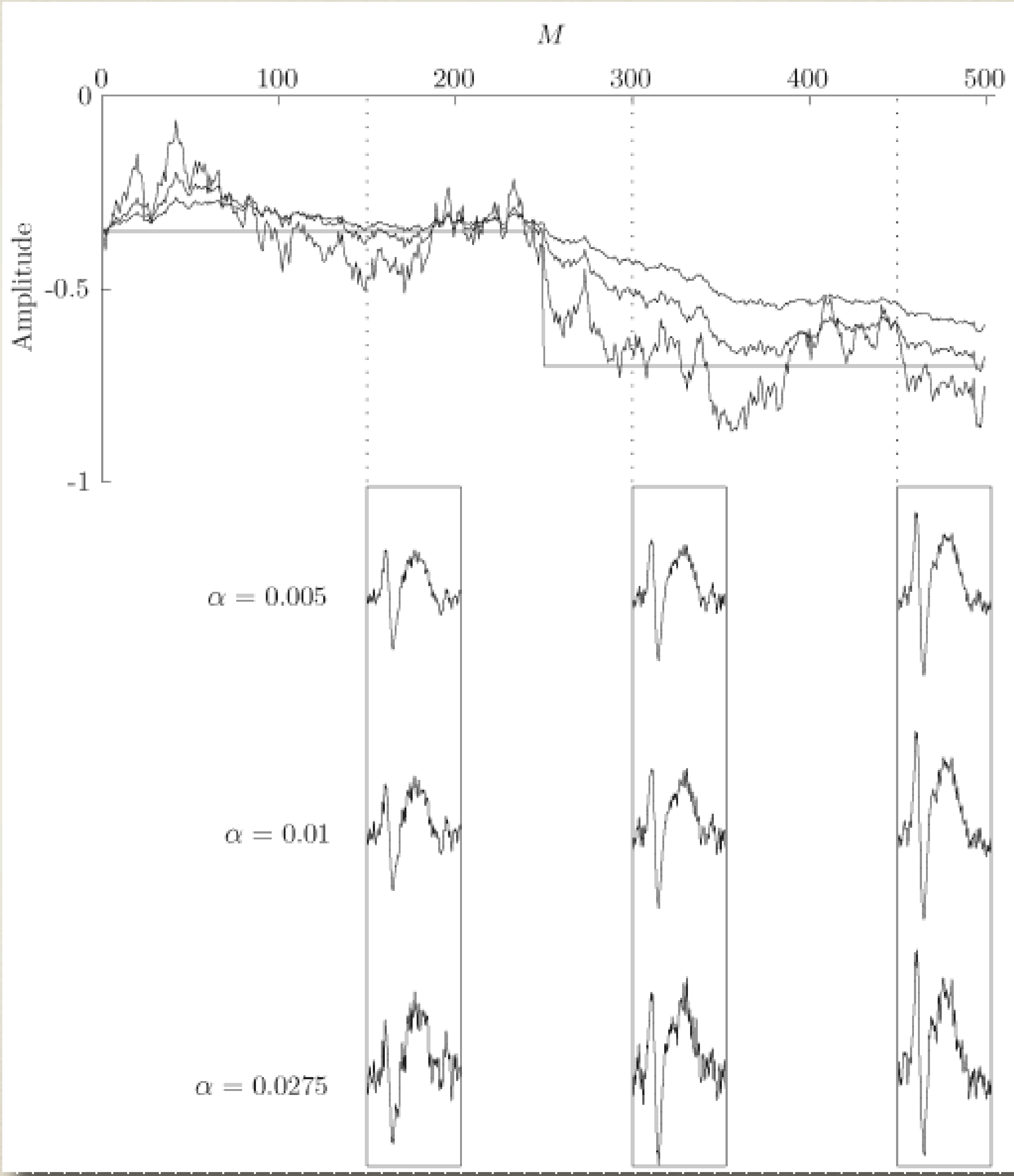
assuming

$$\hat{\mathbf{s}}_{a,0} = \mathbf{0},$$

Exponential averaging results from replacing the weight 1/M with alpha:

$$\hat{\mathbf{s}}_{e,M} = \hat{\mathbf{s}}_{e,M-1} + \alpha(\mathbf{x}_M - \hat{\mathbf{s}}_{e,M-1}).$$

Exponential averaging



Noise Reduction of EPs with Varying Noise Level

- Assumption: all evoked potentials have
  - identical shapes s(n) but with
  - varying noise level.
- Such an heterogenous ensemble is processed by weighted averaging.

Weighted Averaging

The weighted average is obtained by weighting each potential $x_i(n)$ with its inverse noise variance:

$$\hat{s}_w(n) = \frac{1}{\displaystyle\sum_{i=1}^{M} \frac{1}{\sigma_{v_i}^2}} \sum_{i=1}^{M} \frac{x_i(n)}{\sigma_{v_i}^2}.$$

where each weight w_i thus is

$$w_i = \frac{\dfrac{1}{\sigma_{v_i}^2}}{\displaystyle\sum_{j=1}^{M} \frac{1}{\sigma_{v_j}^2}},$$

This expression reduces to the ensemble average when the noise variance is identical in all potentials.

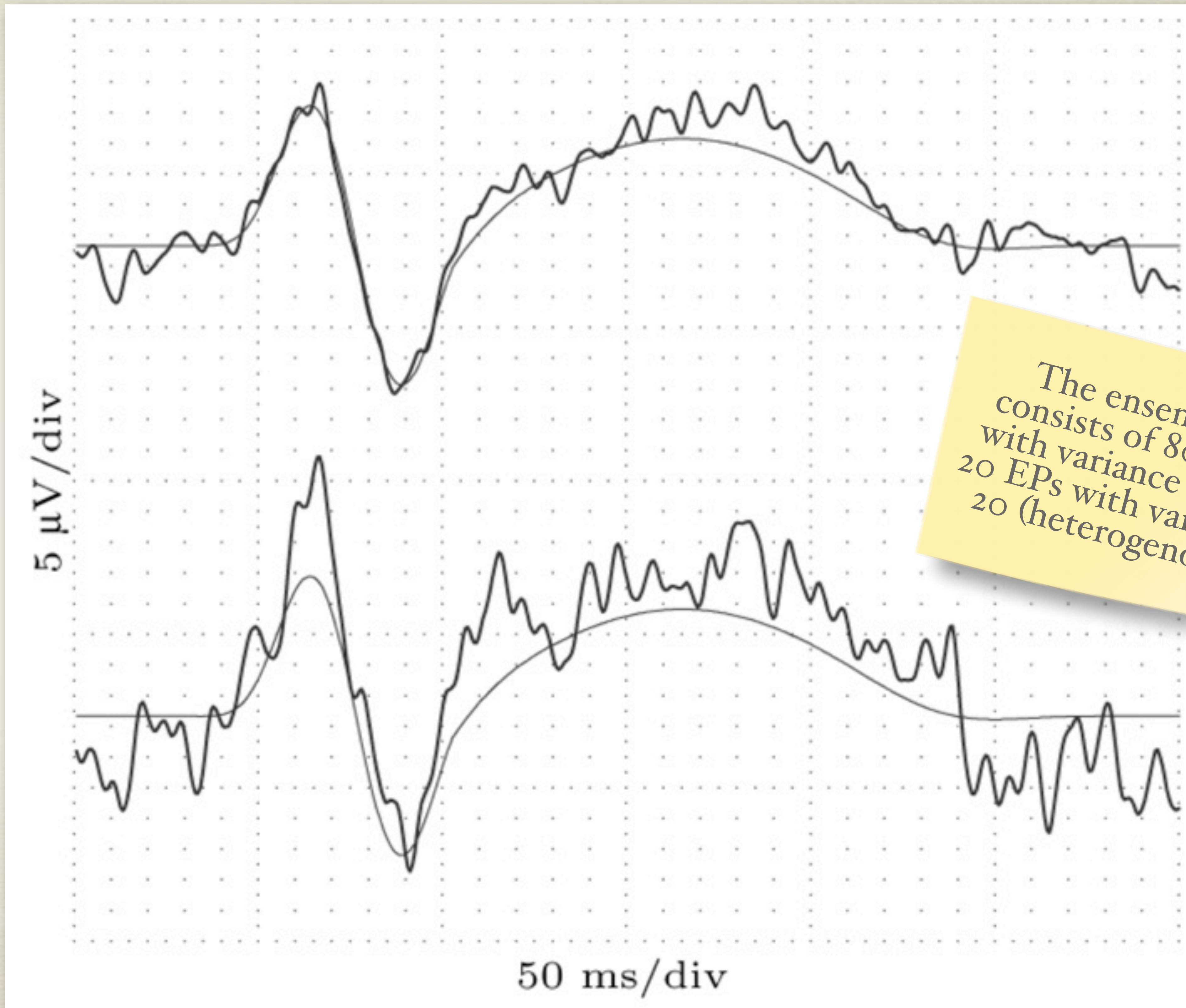
Weighted Averaging, cont'

How to estimate
the varying noise level?

Weighted averaging: An Example

Weighted average

Ensemble average

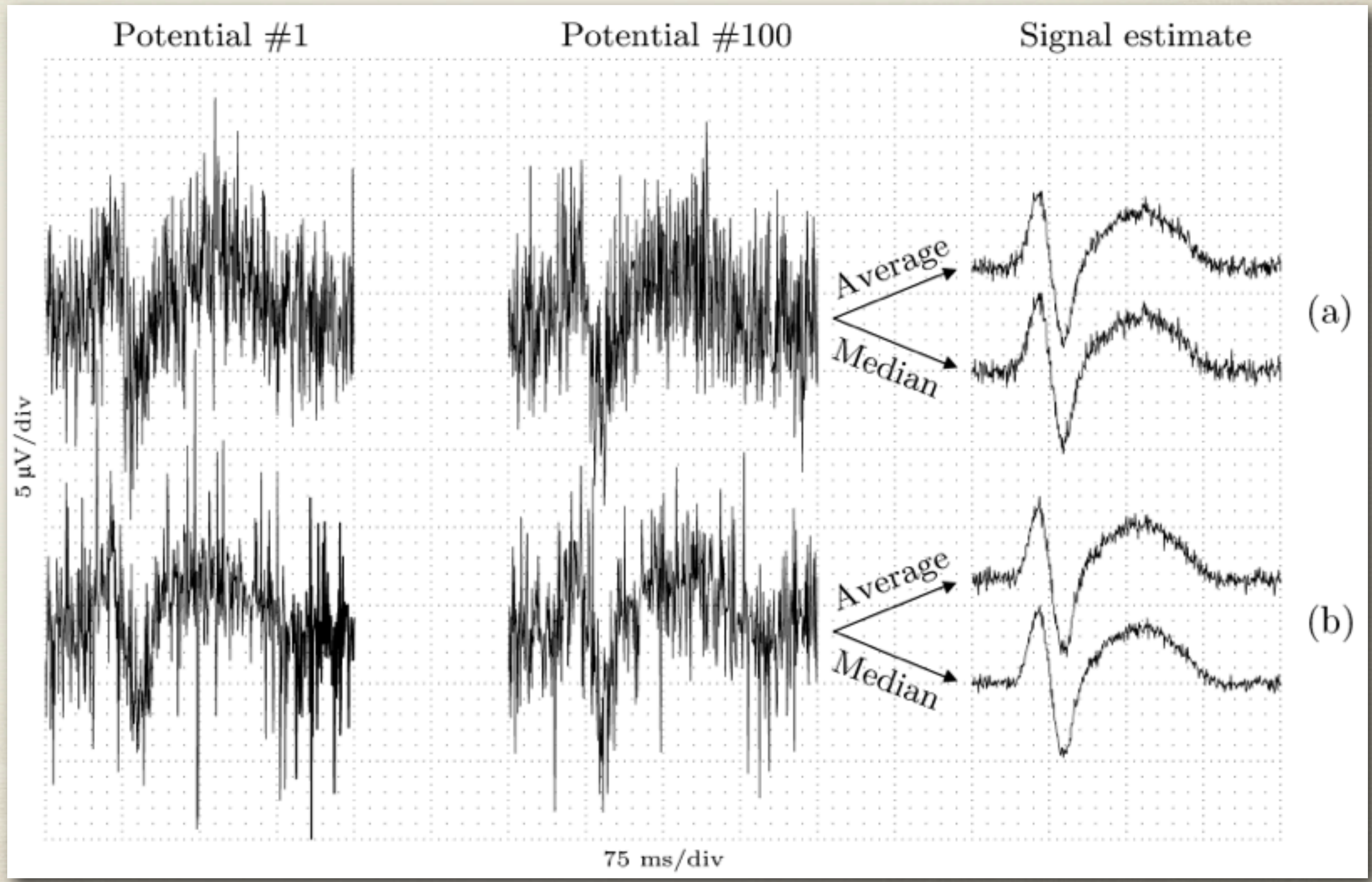


The ensemble consists of 80 EPs with variance 1 and 20 EPs with variance 20 (heterogenous)

Robust Waveform Averaging

Gaussian noise

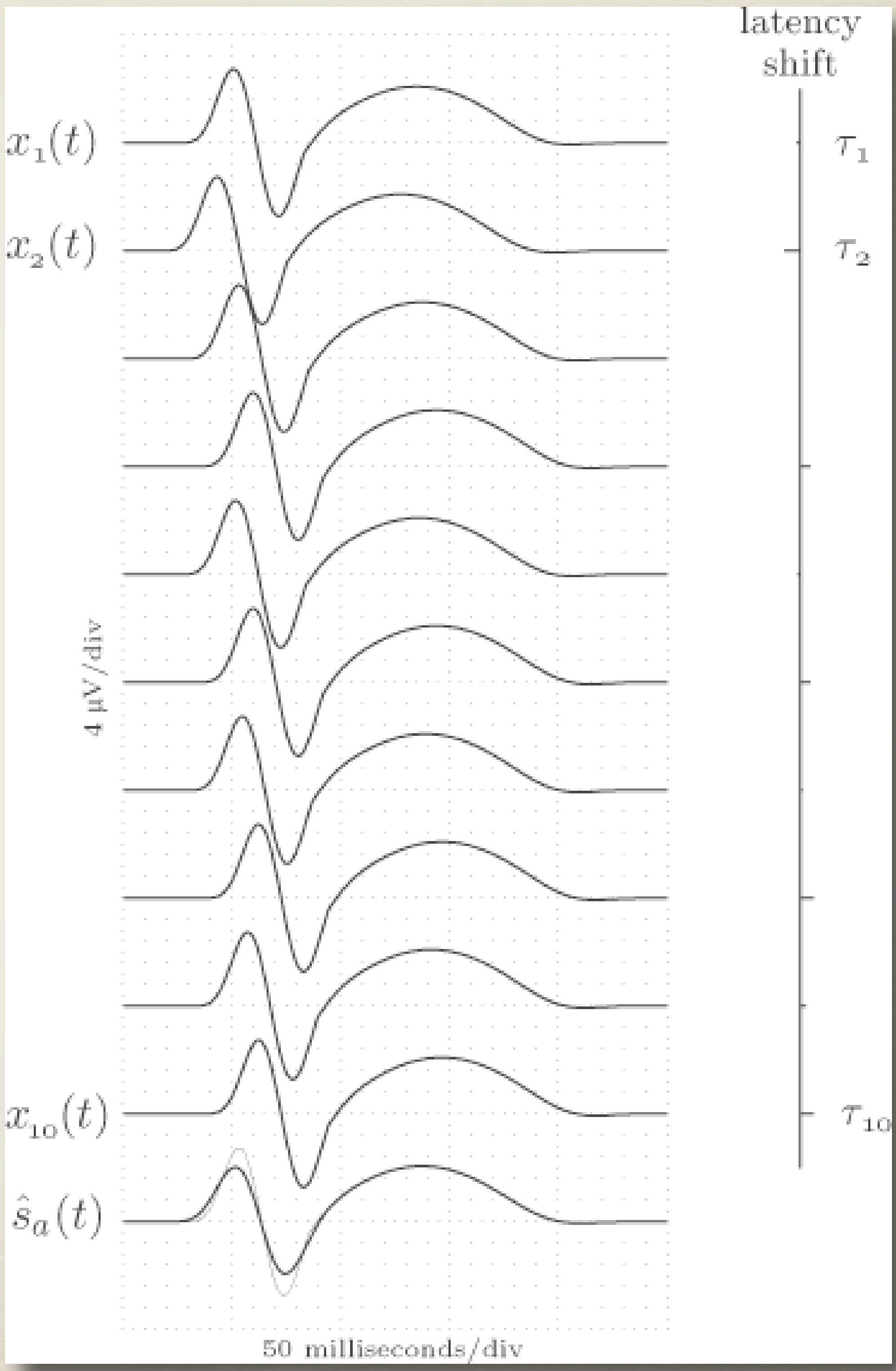
Laplacian noise



The Effect of Latency Variations

Signal model:

$$x_i(n) = s(n - \theta_i) + v_i(n)$$



Lowpass Filtering of the Signal

The expected value of the ensemble average, in the presence of latency variations, is given by:

$$E[\hat{s}_a(t)] = \frac{1}{M}\sum_{i=1}^{M} E[s(t-\tau_i)]$$
$$= \int_{-\infty}^{\infty} s(t-\tau)\ p_\tau(\tau)\ d\tau.$$

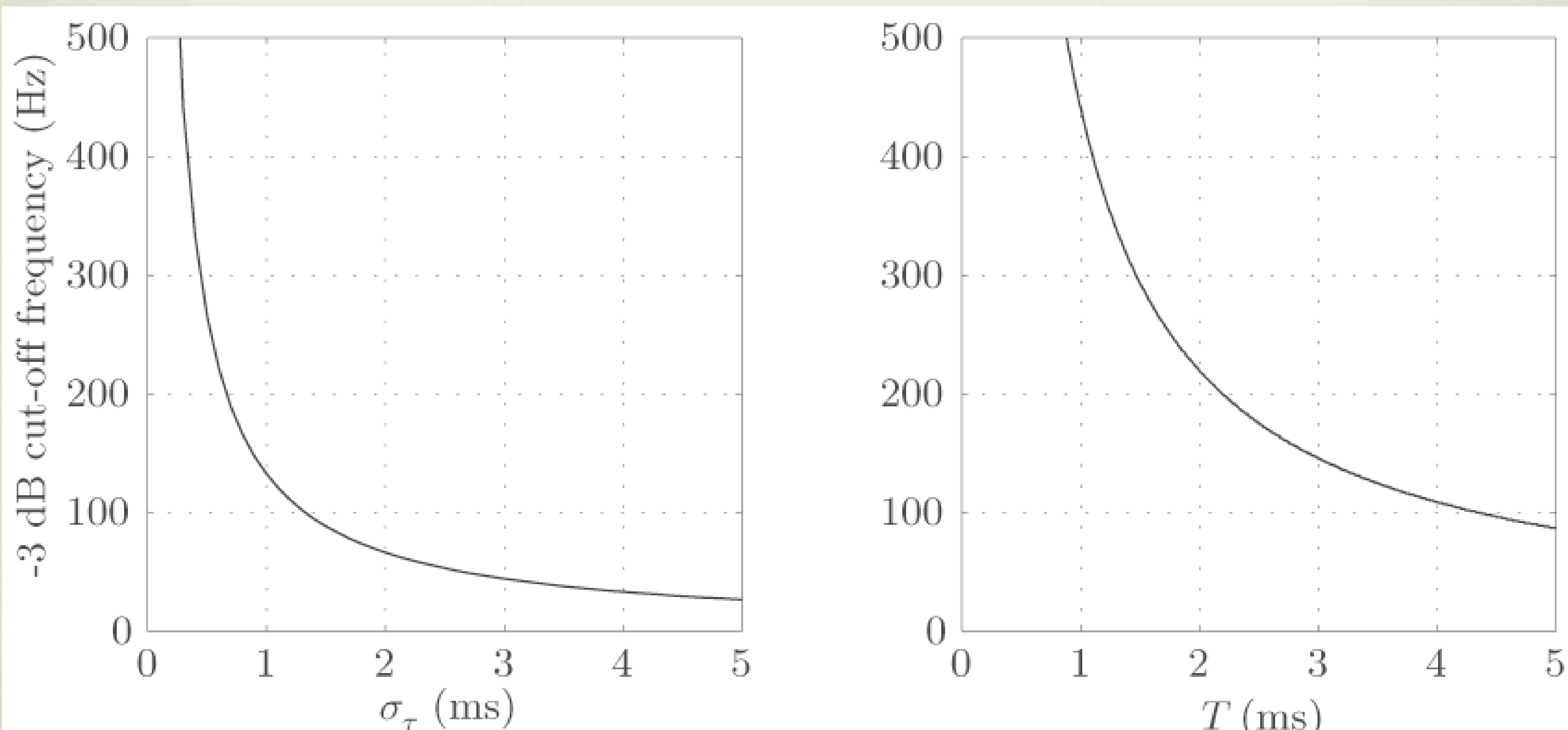
or, equivalently, in the frequency domain:

$$E[\hat{S}_a(\Omega)] = S(\Omega)P_\tau^*(\Omega),$$

Latency Variation and Lowpass Filtering

Gaussian PDF

Uniform PDF



Techniques for Correction of Latency Variations

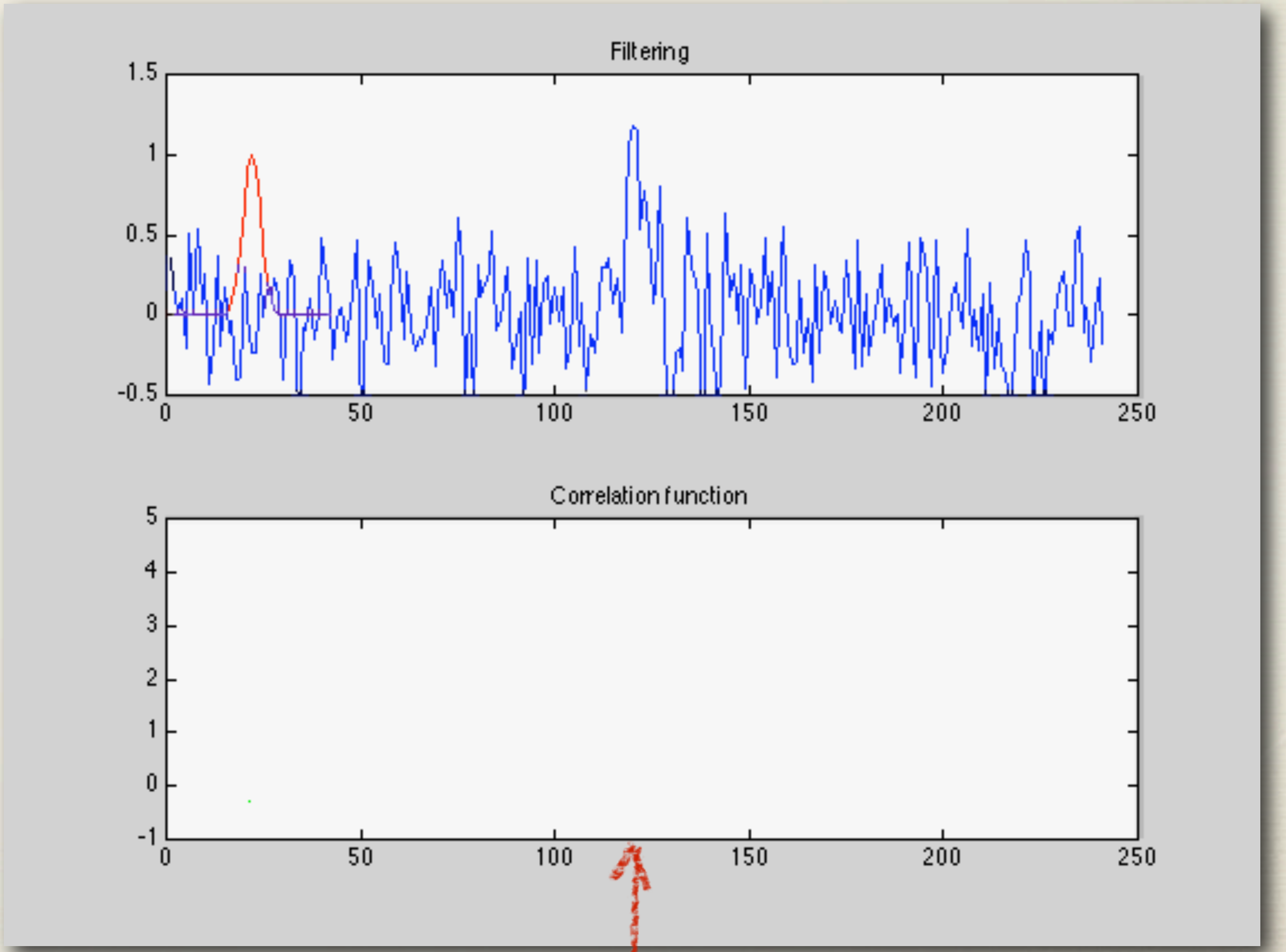
* Synchronize with respect to a peak of the signal or similar property.
* Crosscorrelation between two EPs.
* Woody's method for iterative synchronization of all responses of the ensemble. The method terminates when no further latency corrections are done.

Estimation of Latency
An Illustration

Input signal

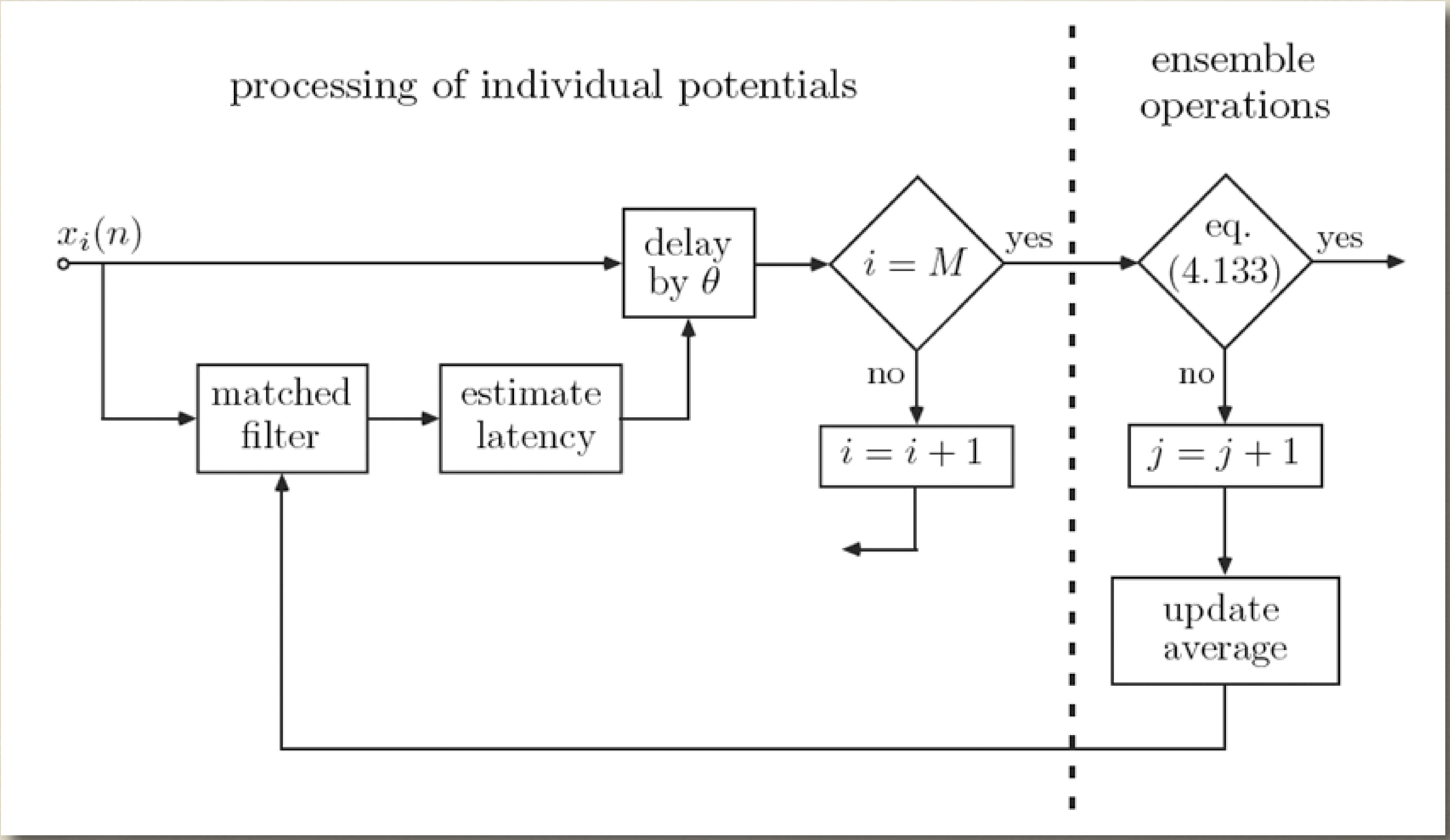
Template waveform

Correlation function



Latency estimate

Woody's Method

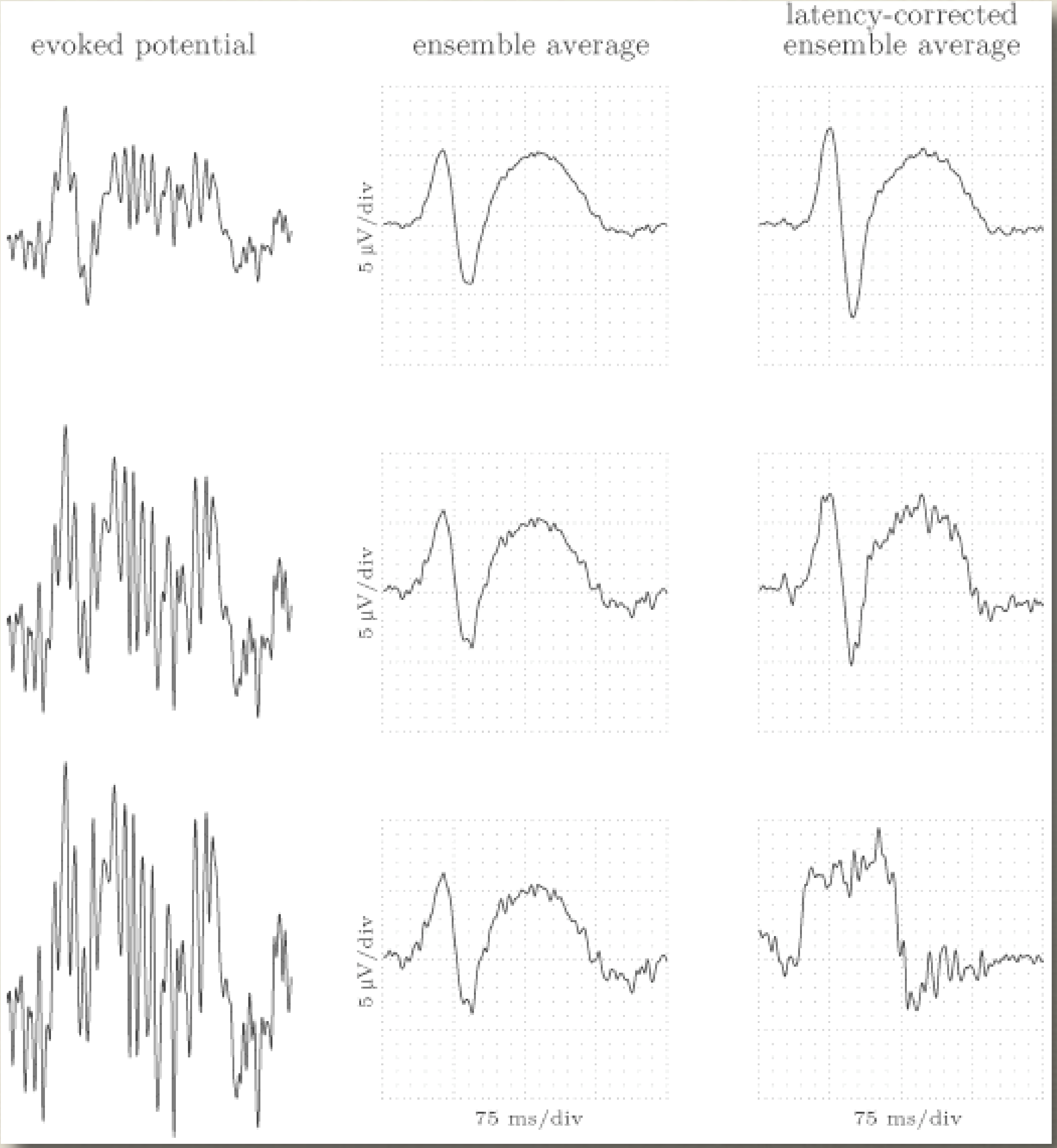


Woody's Method: Different SNRs

good SNR

not so good SNR

bad SNR



SNR-based Weighting

Design a weight function $w(n)$ which minimizes

$$E\left[\left(s(n)-\hat{s}_a(n)w(n)\right)^2\right]$$

where $s(n)$ denotes the desired signal and $\hat{s}_a(n)$ the ensemble average. The optimal "filter" is

$$w(n)=\frac{\sigma_s^2(n)}{\sigma_s^2(n)+\frac{\sigma_v^2}{M}}=\frac{1}{1+\frac{\sigma_v^2}{M\sigma_s^2(n)}}$$

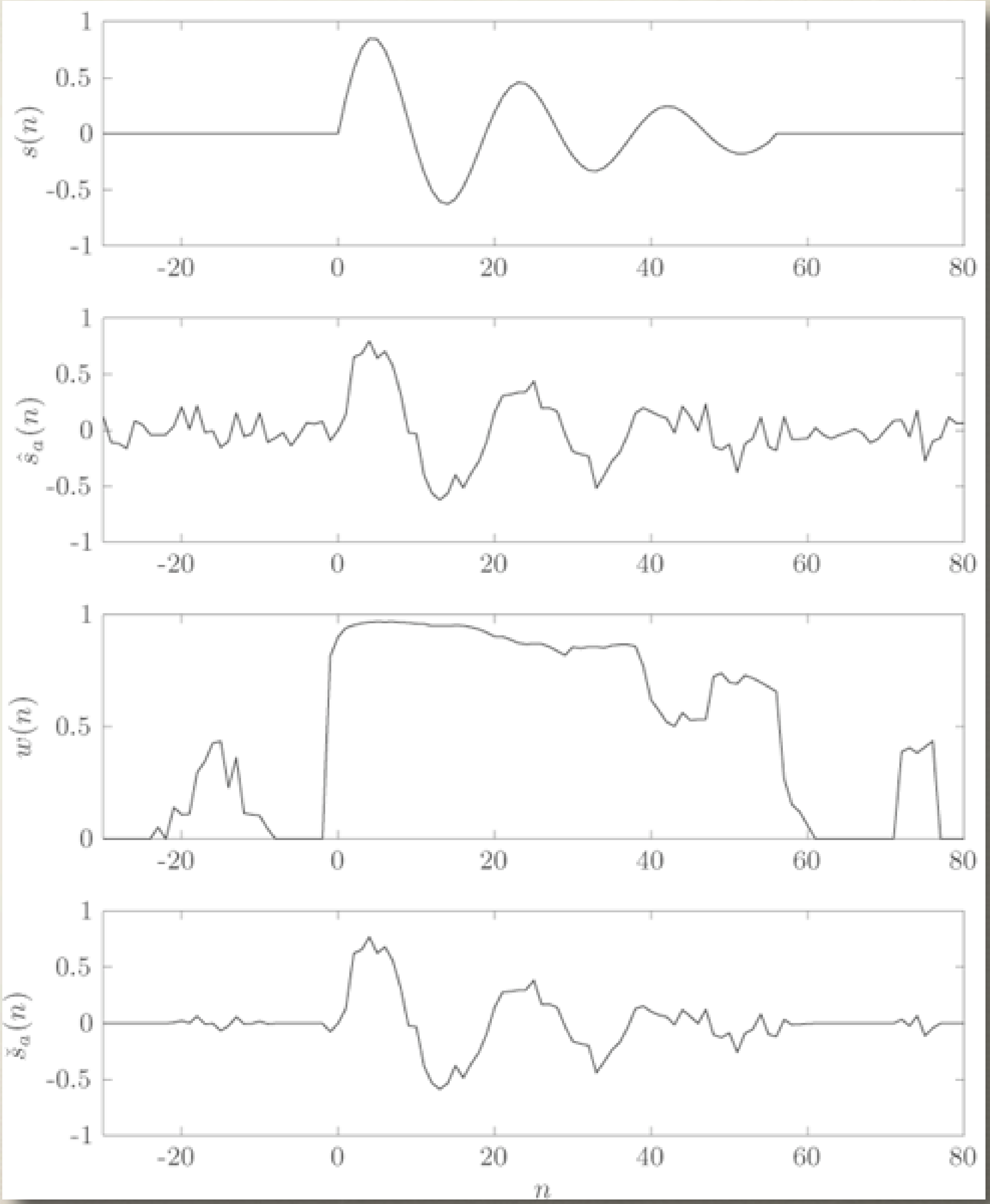
SNR-based Weighting

Noise-free signal

Ensemble average

Weight function

Weight function multiplied with ensemble average



Noise Reduction by Filtering

* Estimate the signal and noise power spectra from the ensemble of signals.
* Design a linear, time-invariant, linear filter such that the mean square error is minimized, i.e., design a Wiener filter.
* Apply the Wiener filter to the ensemble average to improve its SNR.

Wiener Filtering

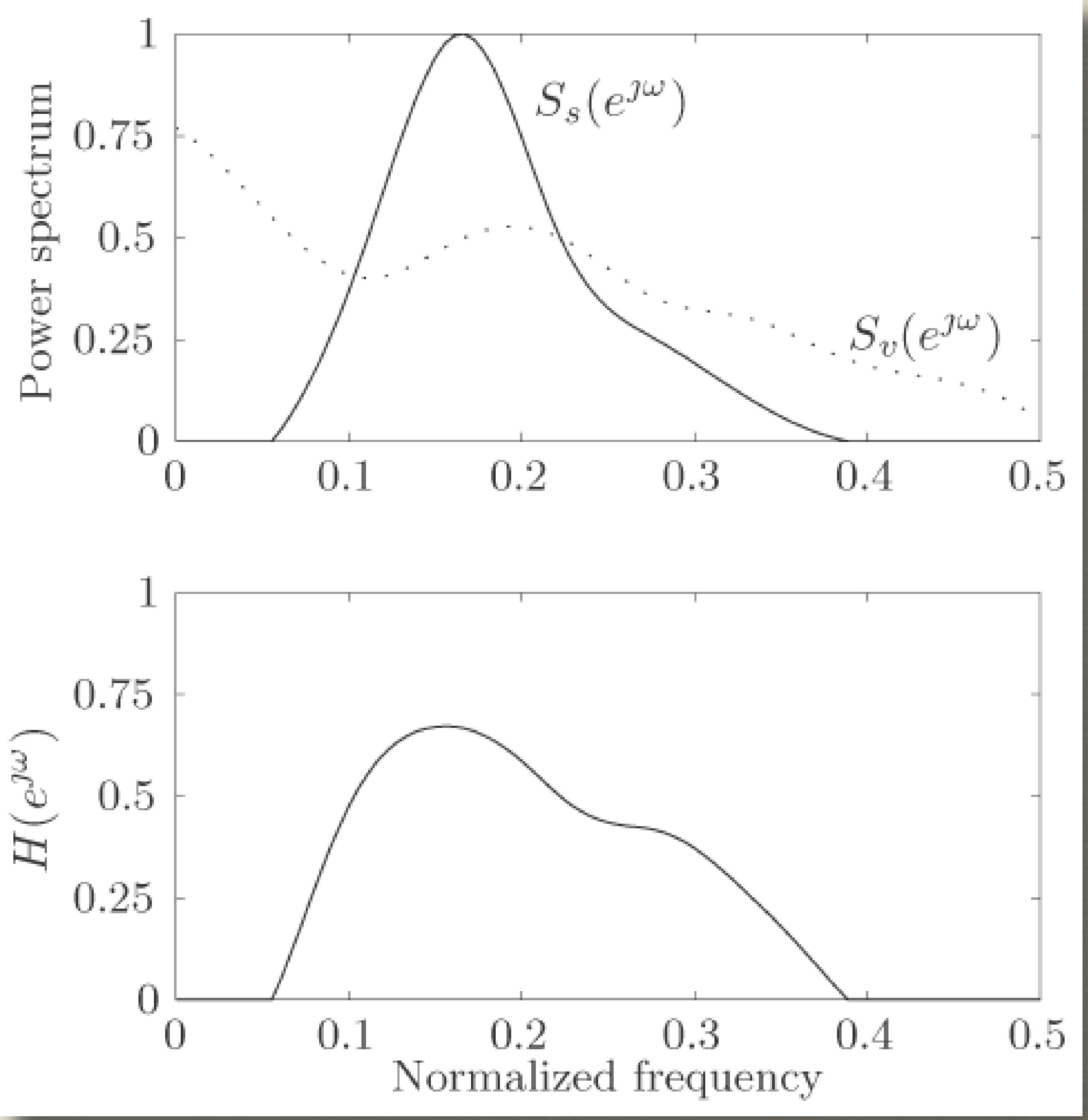
$S_s(e^{j\omega})$: signal power spectrum

$S_v(e^{j\omega})$: noise power spectrum

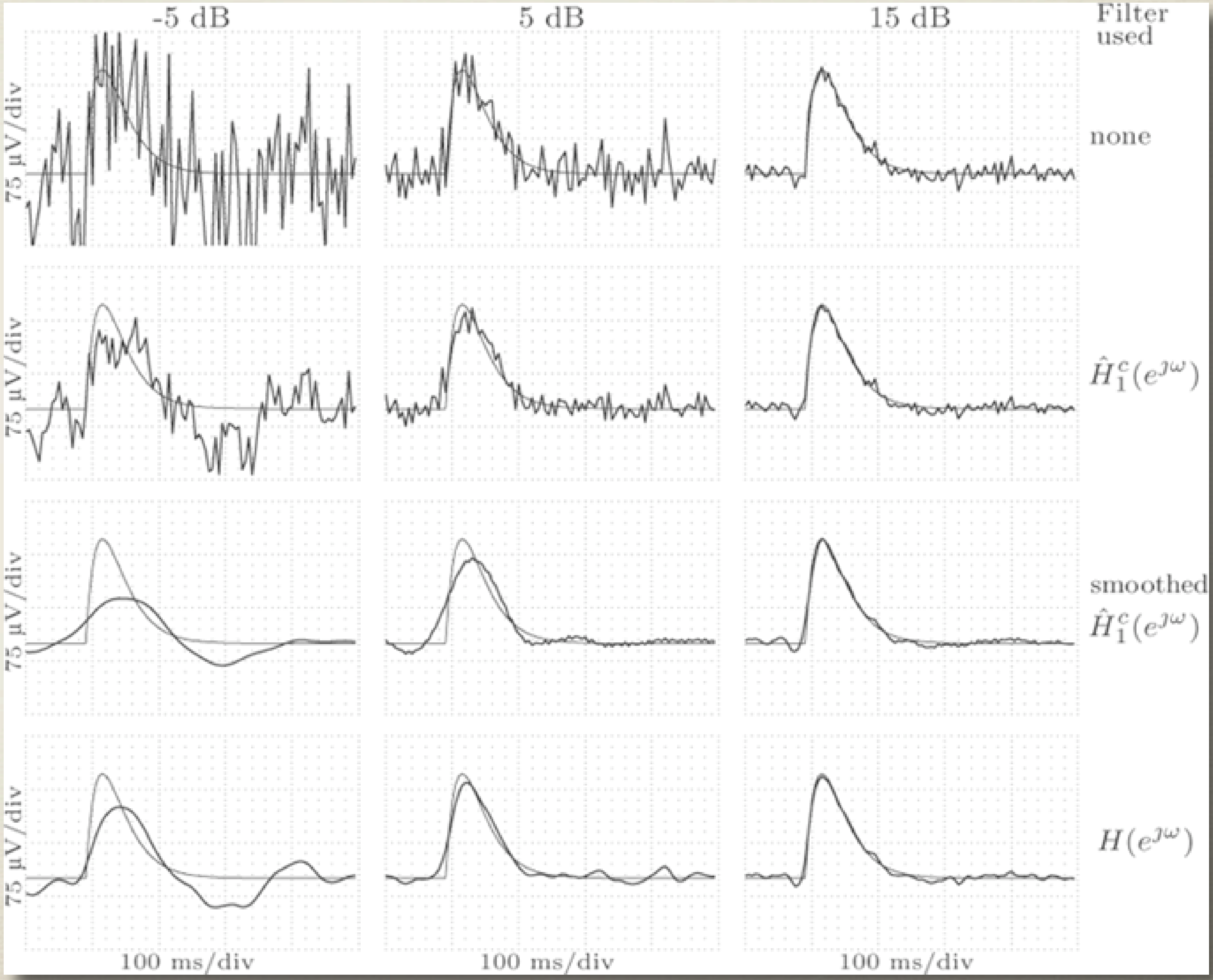
Wiener filter:

$$H(e^{j\omega}) = \frac{S_s(e^{j\omega})}{S_s(e^{j\omega}) + S_v(e^{j\omega})}$$

for one potential
for M potentials



Filtering of Evoked Potentials



Limitations of Wiener filtering

* Assumes that the observed signal is stationary (which in practice it is not...).
* Filtering causes the EP peak amplitudes to be severely underestimated at low SNRs.
* As a result, this technique is rarely used in practice.

Tracking of EP Morphology

* So far, noise reduction has been based on the entire ensemble, e.g., weighted or exponential averaging
* We will now track changes in EP morphology by so-called single-sweep analysis. More a priori information is introduced by describing each EP by a set of basis functions.

Selection of Basis Functions

* Orthonormality is an important function property of basis functions.
* Sines/cosines are well-known basis functions, but it is often better to use...
* ...functions especially determined for optimal (MSE) representation of different waveform morphologies (the Karhunen-Loève representation).

Orthogonal Expansions

An EP $\mathbf{x}_i$, composed of both signal and noise, is modeled as a stochastic process which can be represented by a linear combination (series expansion) of basis functions $\boldsymbol{\varphi}_k$,

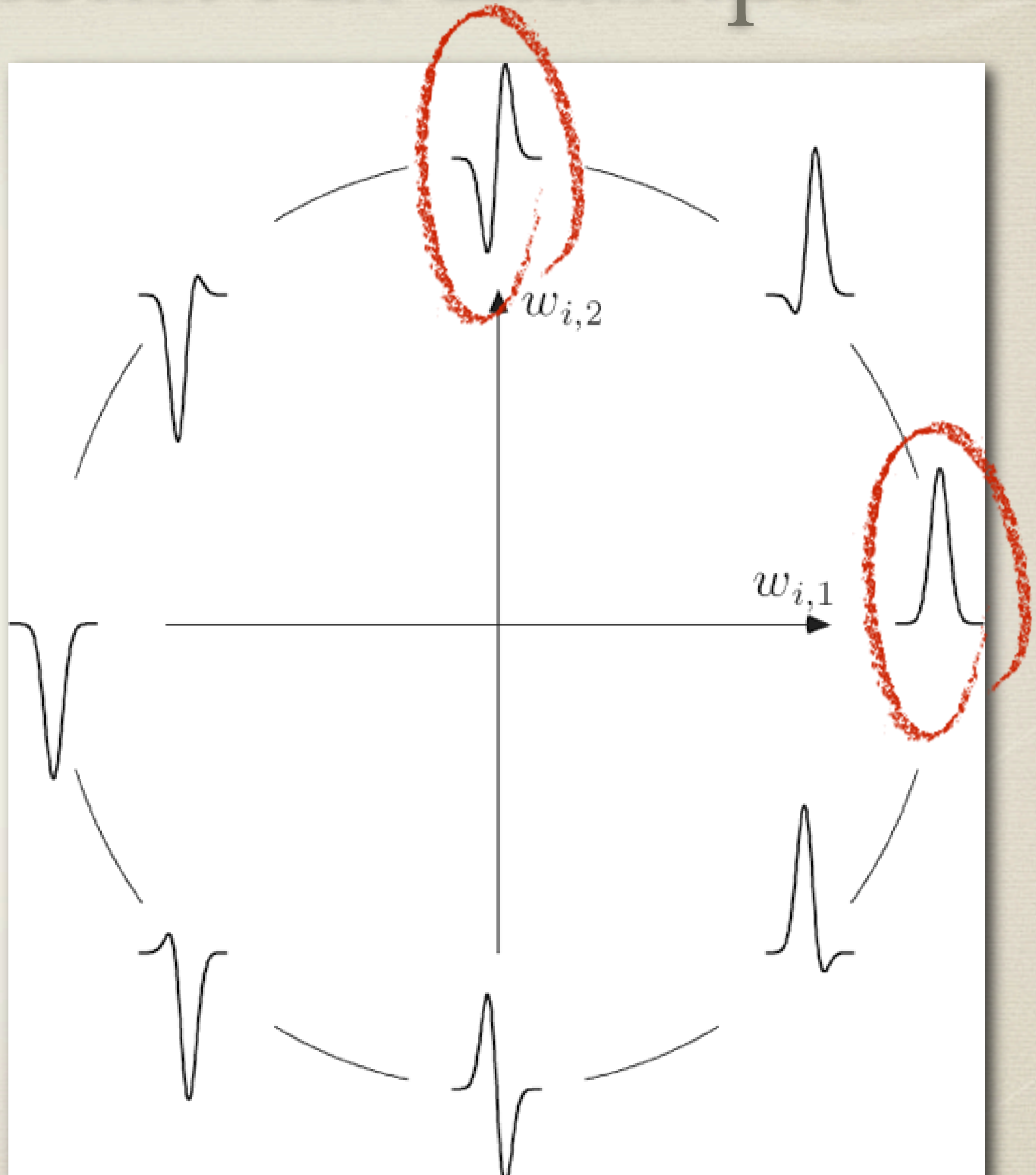
$$\mathbf{x}_i = \sum_{k=1}^{N} w_{i,k}\boldsymbol{\varphi}_k, \tag{4.187}$$

where each basis function is represented by a vector with N elements

$$\boldsymbol{\varphi}_k = \begin{bmatrix} \varphi_k(0) \\ \varphi_k(1) \\ \vdots \\ \varphi_k(N-1) \end{bmatrix}, \quad k = 1, \ldots, N. \tag{4.188}$$

Basis Functions: An Example

Linear combinations of two basis functions model a variety of signal morphologies



Calculation of the Weights

Introducing the matrix notation $\mathbf{\Phi}$ to represent the set of basis functions,

$$\mathbf{\Phi} = \begin{bmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_N \end{bmatrix}, \tag{4.191}$$

we can write the series expansion in (4.187) more compactly as

$$\mathbf{x}_i = \mathbf{\Phi}\mathbf{w}_i. \tag{4.192}$$

The orthonormality property implies that the basis functions are mutually orthogonal, and with their energy normalized to one,

$$\varphi_k^T \varphi_l = \begin{cases} 1, & k = l; \\ 0, & k \neq l. \end{cases} \tag{4.193}$$

the coefficient vector $\mathbf{w}_i$ can be calculated from $\mathbf{x}_i$ using the relation

$$\mathbf{w}_i = \mathbf{\Phi}^T \mathbf{x}_i.$$

$$w_{i,k} = \varphi_k^T \mathbf{x}_i = \sum_{n=0}^{N-1} \varphi_k(n) x_i(n)$$

Mean-Square Weight Estimation

The calculation of $\mathbf{w}_i$ can be treated from an estimation point of view in which $\mathbf{w}_i$ is chosen such that the MSE is minimized. Each EP $\mathbf{x}_i$ is modeled by

$$\mathbf{x}_i = \mathbf{s}_i + \mathbf{v}_i, \tag{4.196}$$

where $E[\mathbf{x}_i] = \mathbf{s}_i$ since the noise $\mathbf{v}_i$ is assumed to be zero-mean. The correlation matrix for the i^{th} EP is

$$\mathbf{R}_{x_i} = E\left[\mathbf{x}_i \mathbf{x}_i^T\right]. \tag{4.197}$$

A suitable criterion to minimize would be the following MSE,

$$E\left[\|\mathbf{s}_i - \mathbf{\Phi}\mathbf{w}_i\|^2\right] = E\left[(\mathbf{s}_i - \mathbf{\Phi}\mathbf{w}_i)^T (\mathbf{s}_i - \mathbf{\Phi}\mathbf{w}_i)\right], \tag{4.198}$$

$$\hat{\mathbf{w}}_i = \mathbf{\Phi}^T \mathbf{x}_i,$$

i.e. identical to the previous expression

Truncated Expansion

The underlying idea of signal estimation through a truncated series expansion is that a subset of basis functions can provide an adequate representation of the signal part.

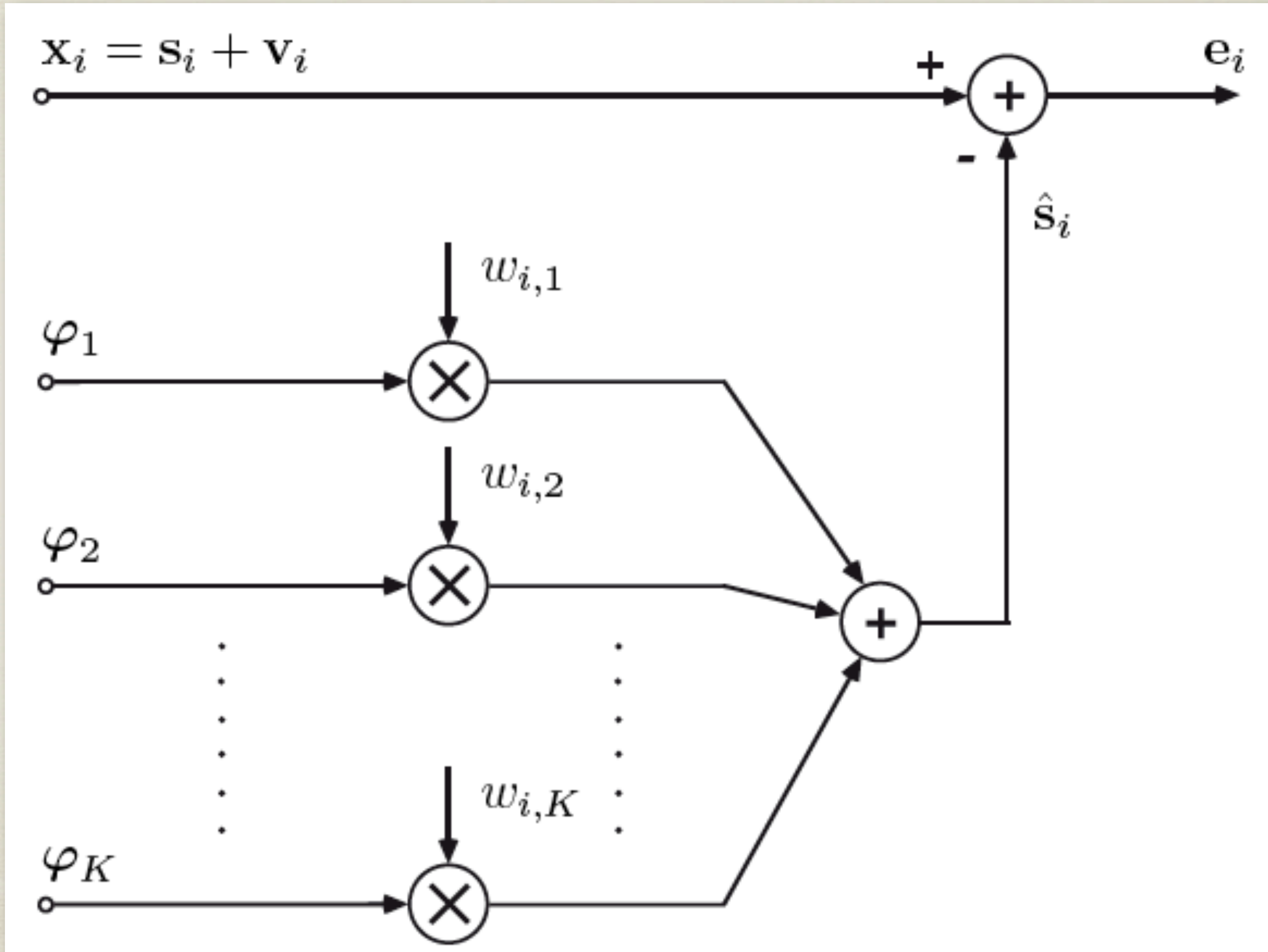
Decomposition into "signal" and "noise" parts:

$$\mathbf{\Phi} = \left[\mathbf{\Phi}_s \quad \mathbf{\Phi}_v\right],$$

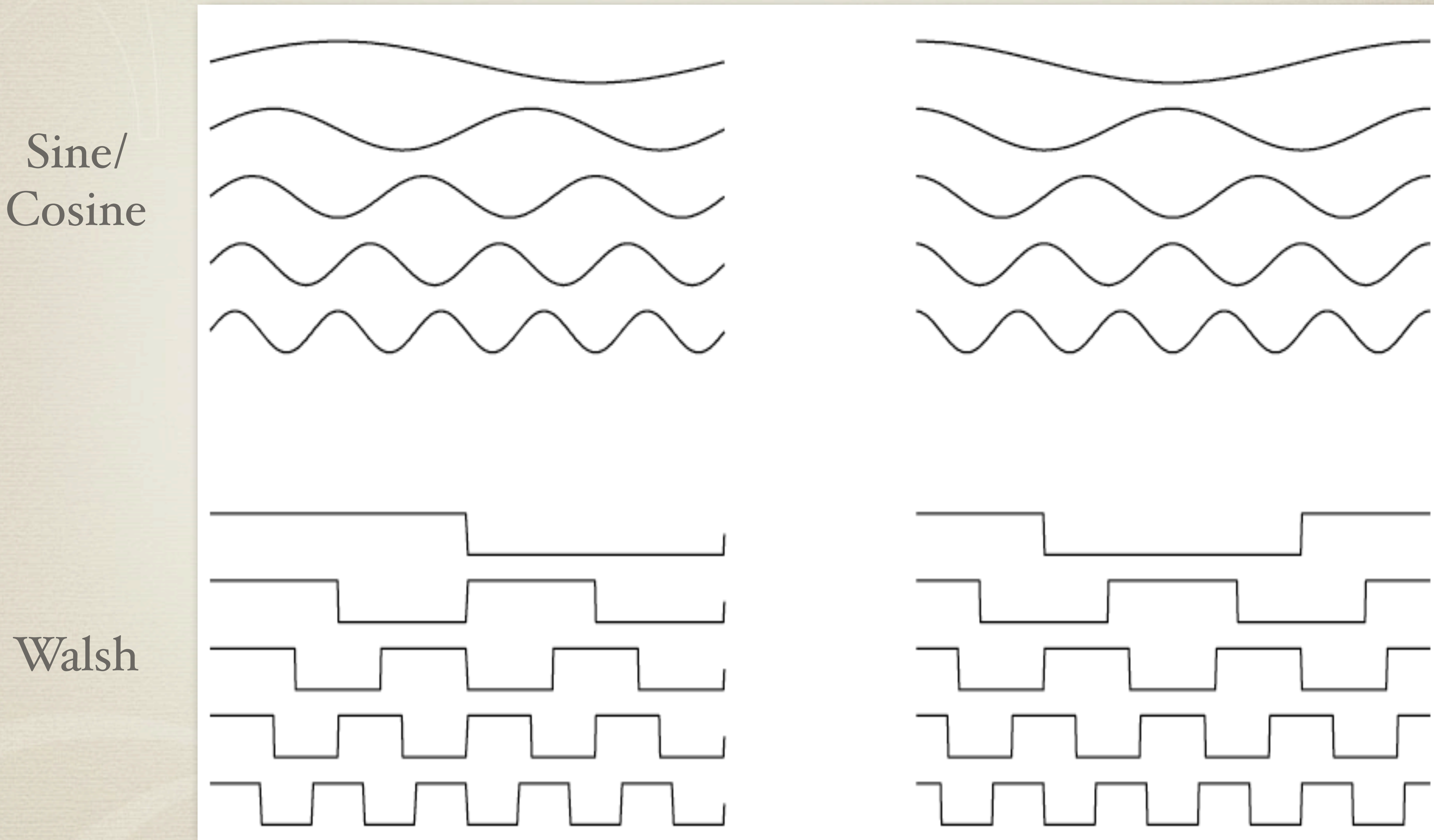
The estimate of the signal is obtained from:

$$\hat{\mathbf{s}}_i = \mathbf{\Phi}_s \hat{\mathbf{w}}_i = \mathbf{\Phi}_s \mathbf{\Phi}_s^T \mathbf{x}_i.$$

Truncated Expansion, cont'



Examples of Basis Functions



Sine/Cosine Modeling

#basis functions

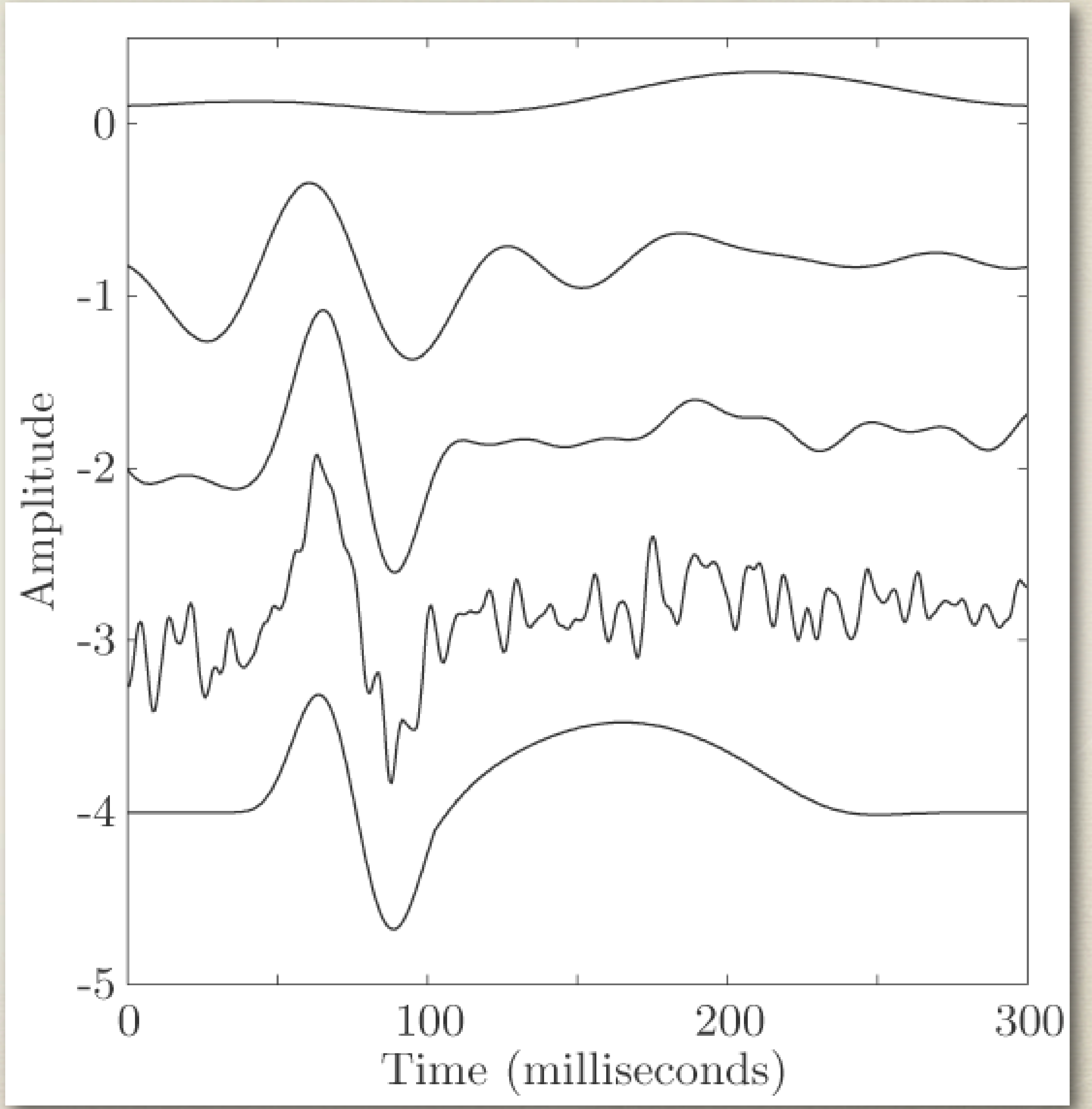
K = 3

K = 7

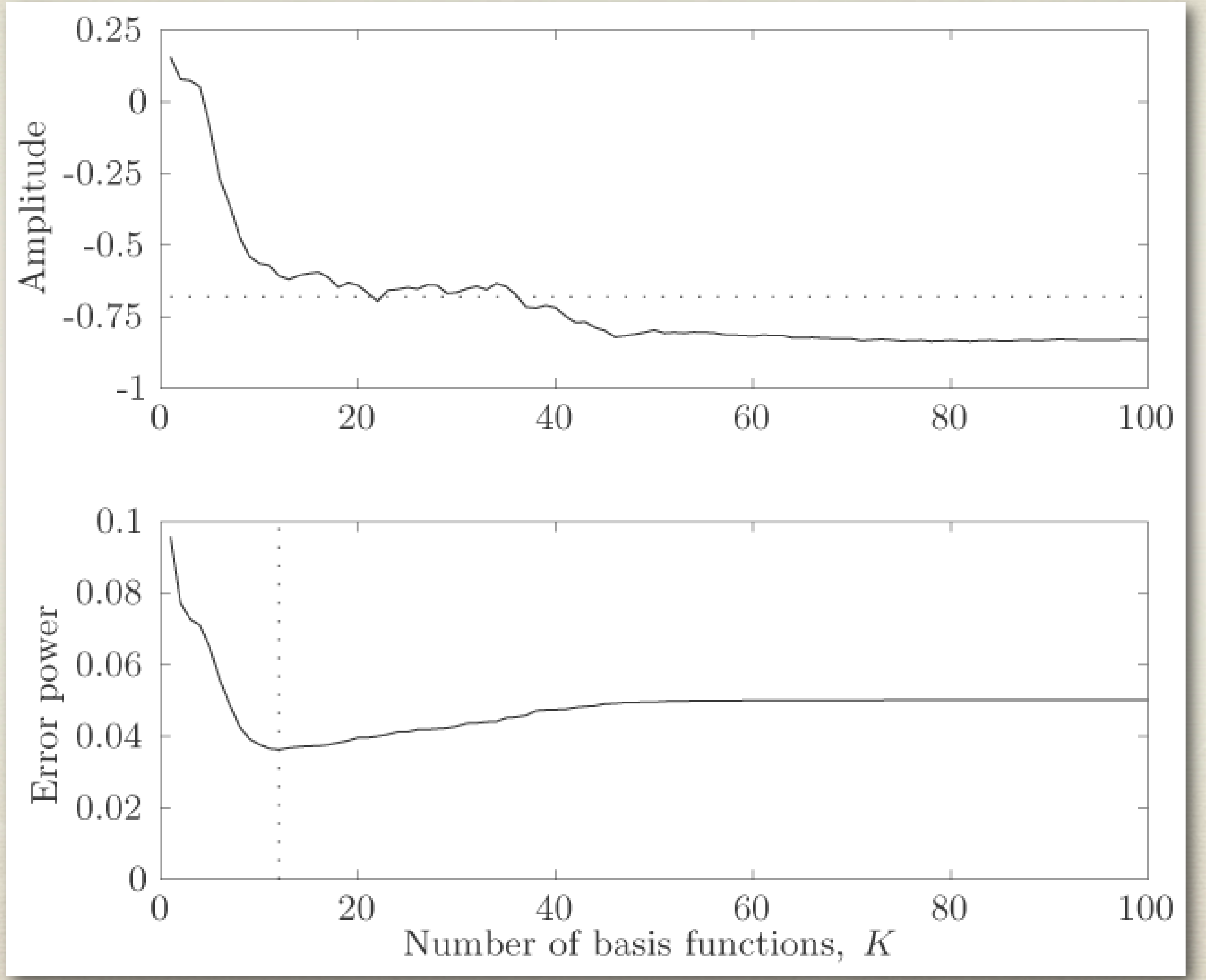
K = 12

K = 500

VEP without noise



Sine/Cosine Modeling: Amplitude Estimate and MSE Error



MSE Basis Functions

How should the basis functions be designed so that the signal part is efficiently represented with a small number of functions?

We start our derivation by decomposing the series expansion of the signal into two sums, that is,

$$\mathbf{x} = \sum_{k=1}^{K} w_k \boldsymbol{\varphi}_k + \sum_{k=K+1}^{N} w_k \boldsymbol{\varphi}_k = \hat{\mathbf{s}} + \hat{\mathbf{v}}, \tag{4.223}$$

where the K first basis functions produce an estimate of $\mathbf{s}$, and the remaining $(N-K)$ terms produce the noise estimate $\hat{\mathbf{v}}$. Our aim is now to find the set of $\boldsymbol{\varphi}_k$'s that makes $\hat{\mathbf{s}}$ resemble $\mathbf{s}$ as closely as possible. This objective can be achieved by minimizing the noise power estimate in the MSE sense,

$$\mathcal{E} = E\left[\hat{\mathbf{v}}^T\hat{\mathbf{v}}\right] = E\left[(\mathbf{x}-\hat{\mathbf{s}})^T(\mathbf{x}-\hat{\mathbf{s}})\right], \tag{4.224}$$

Karhunen–Loève Basis Functions

The Karhunene–Loève (KL) basis functions, minimizing the MSE, are obtained as the solution of the ordinary eigenvalue problem, and equals the eigenvectors corresponding to the largest eigenvalues:

$$\mathbf{R}_x \boldsymbol{\varphi}_k = \lambda_k \boldsymbol{\varphi}_k,$$

The MSE equals the sum of the (N-K) smallest eigenvalues

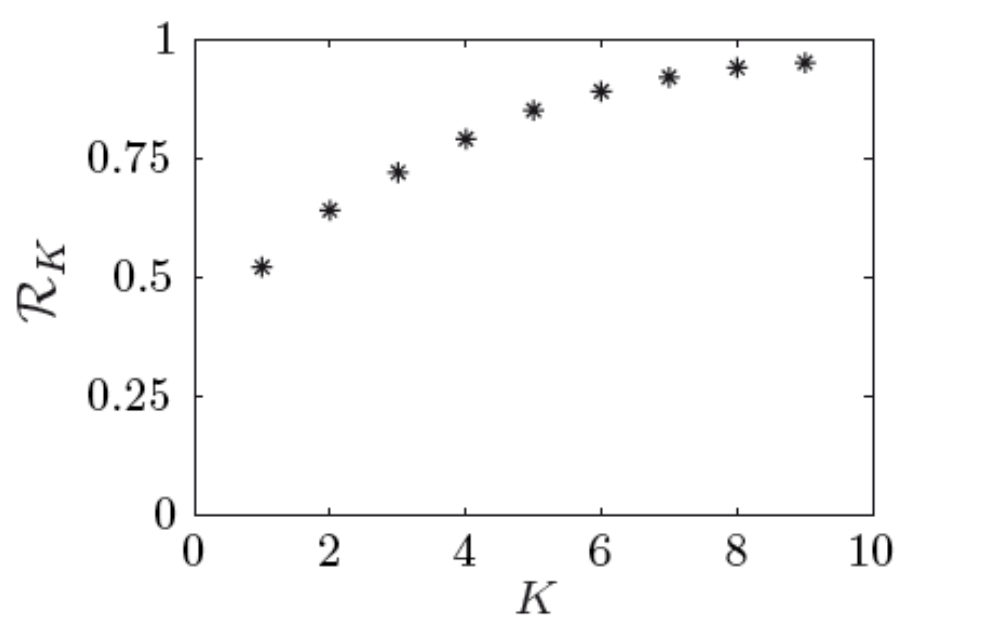
$$\mathcal{E} = \sum_{k=K+1}^{N} \lambda_k$$

KL Performance Index

Given an ensemble of signals characterized by $\mathbf{R}_x$, a performance index $\mathcal{R}_K$ can be defined which reflects how well the truncated series expansion approximates the ensemble in energy terms,

$$\mathcal{R}_K = \frac{\sum_{k=1}^{K} \lambda_k}{\sum_{k=1}^{N} \lambda_k}. \tag{4.243}$$

Example of the performance index



How to get $\mathbf{R}_x$?

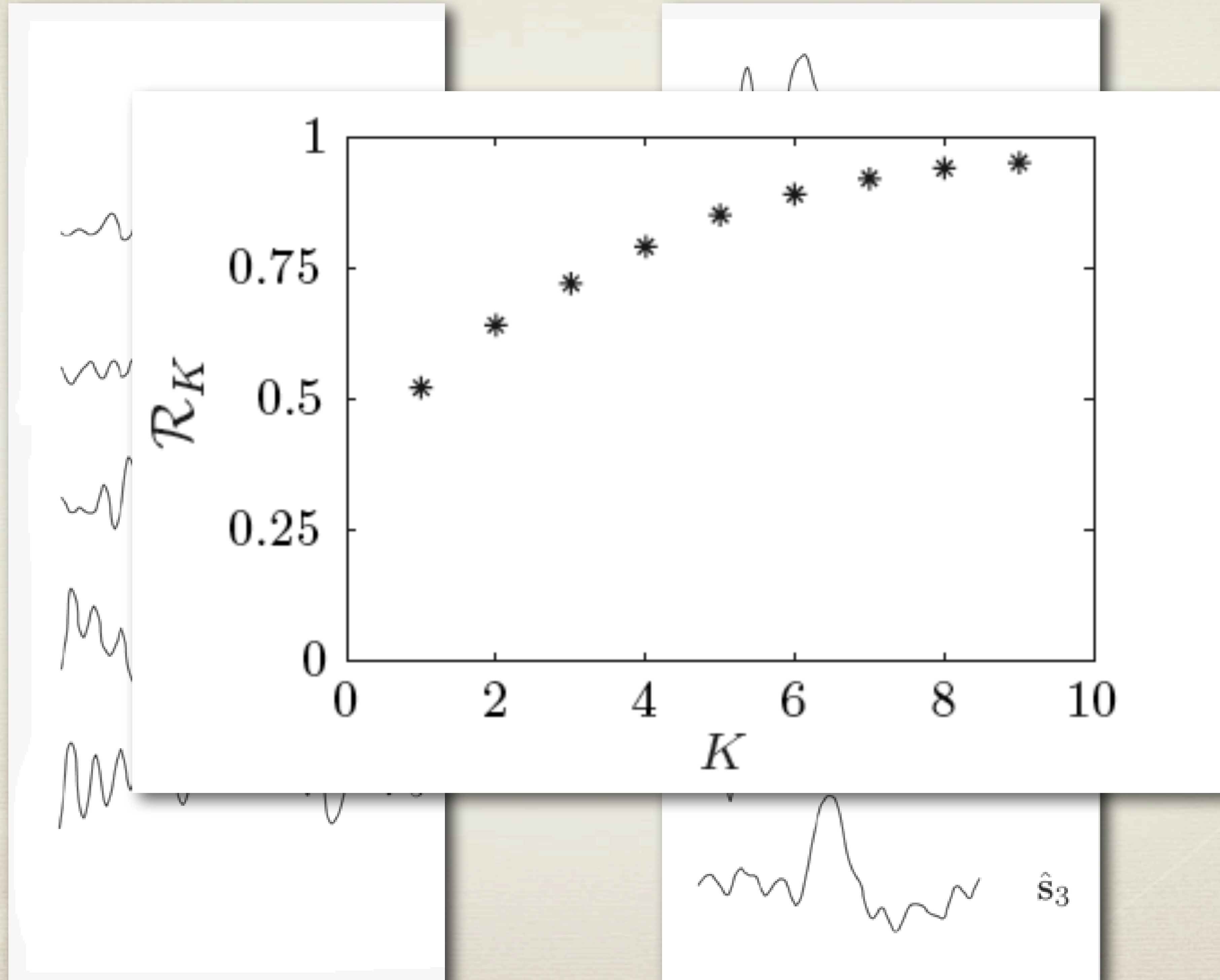
In practice, the correlation matrix $\mathbf{R}_x$ cannot be estimated from a single potential but must be estimated from the ensemble $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_M$. The estimation of $\mathbf{R}_x$ is commonly achieved by simply replacing the expected value in the definition of $\mathbf{R}_x$ (cf. (3.6)) by averaging the M rank-one correlation matrices $\mathbf{x}_i\mathbf{x}_i^T$ for each of the EPs,

$$\hat{\mathbf{R}}_x = \frac{1}{M}\sum_{i=1}^{M}\mathbf{x}_i\mathbf{x}_i^T. \tag{4.244}$$

Example: KL Basis Functions

Basis functions

Signals



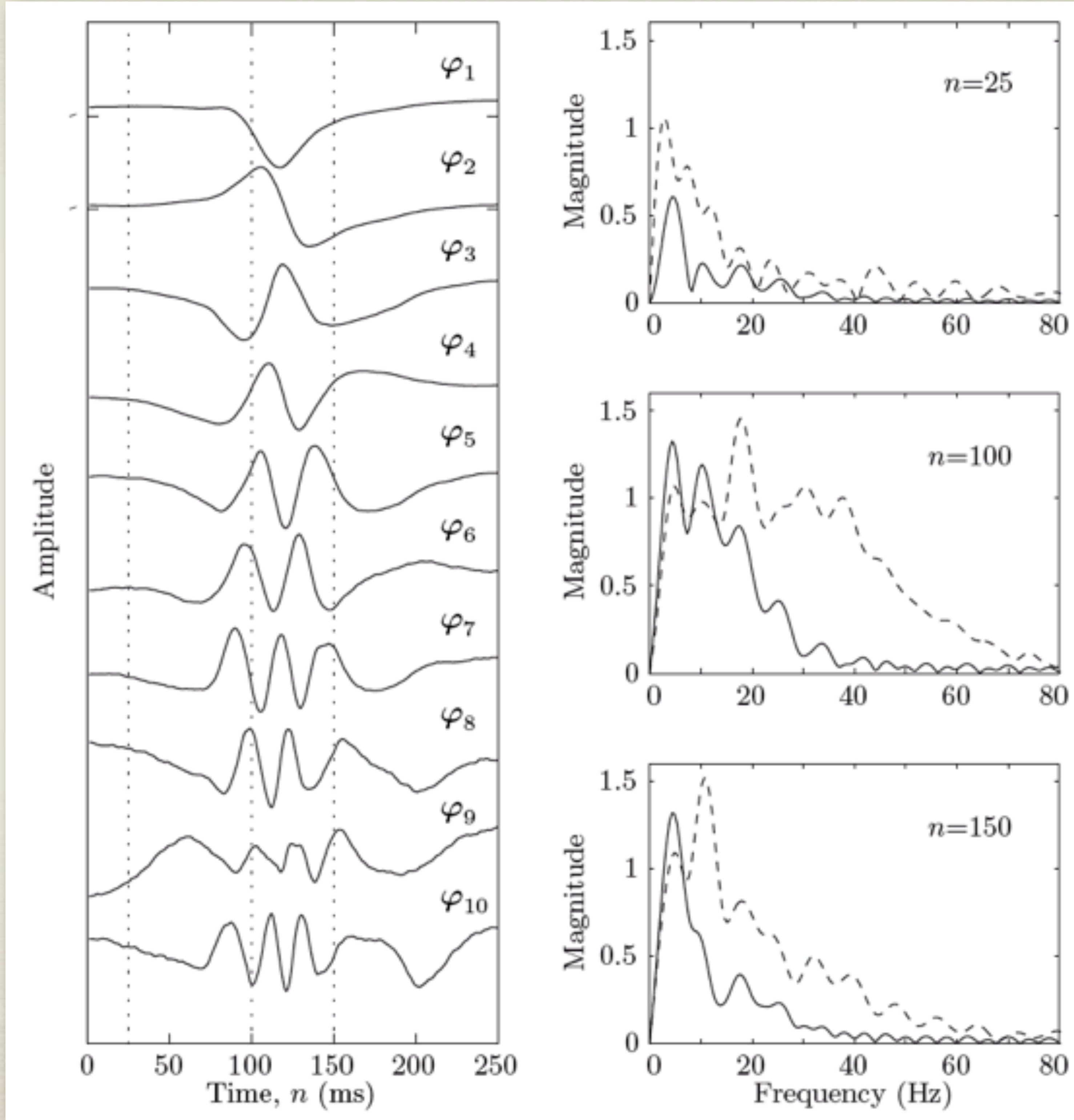
bserved

gnal: $\mathbf{x}_i$

gnal

timate: $\hat{\mathbf{s}}_i$

Time-Varying Filter Interpretation



Modeling with Damped Sinusoids

$$x(n) = \sum_{k=1}^{K} w_k e^{\rho_k n} e^{\jmath(\omega_k n + \phi_k)},$$

for $n = 0, \ldots, N-1$. Each term of the expansion is characterized by its amplitude w_k, frequency ω_k, phase ϕ_k, and a damping factor ρ_k ($\rho_k < 0$) which determines the decay in amplitude.

* The original Prony method
* The least-squares Prony method
* Variations

Adaptive Estimation of Weights

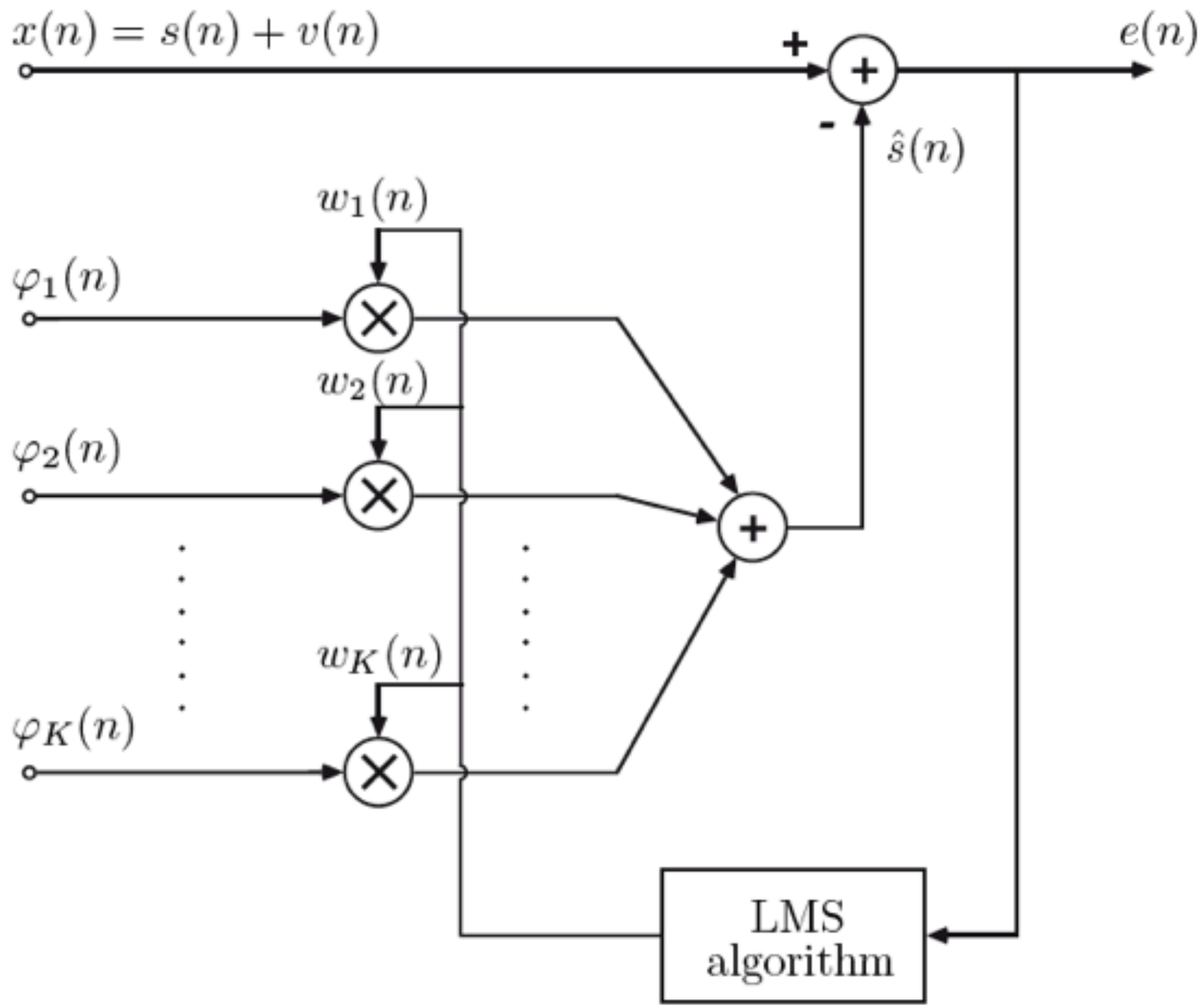
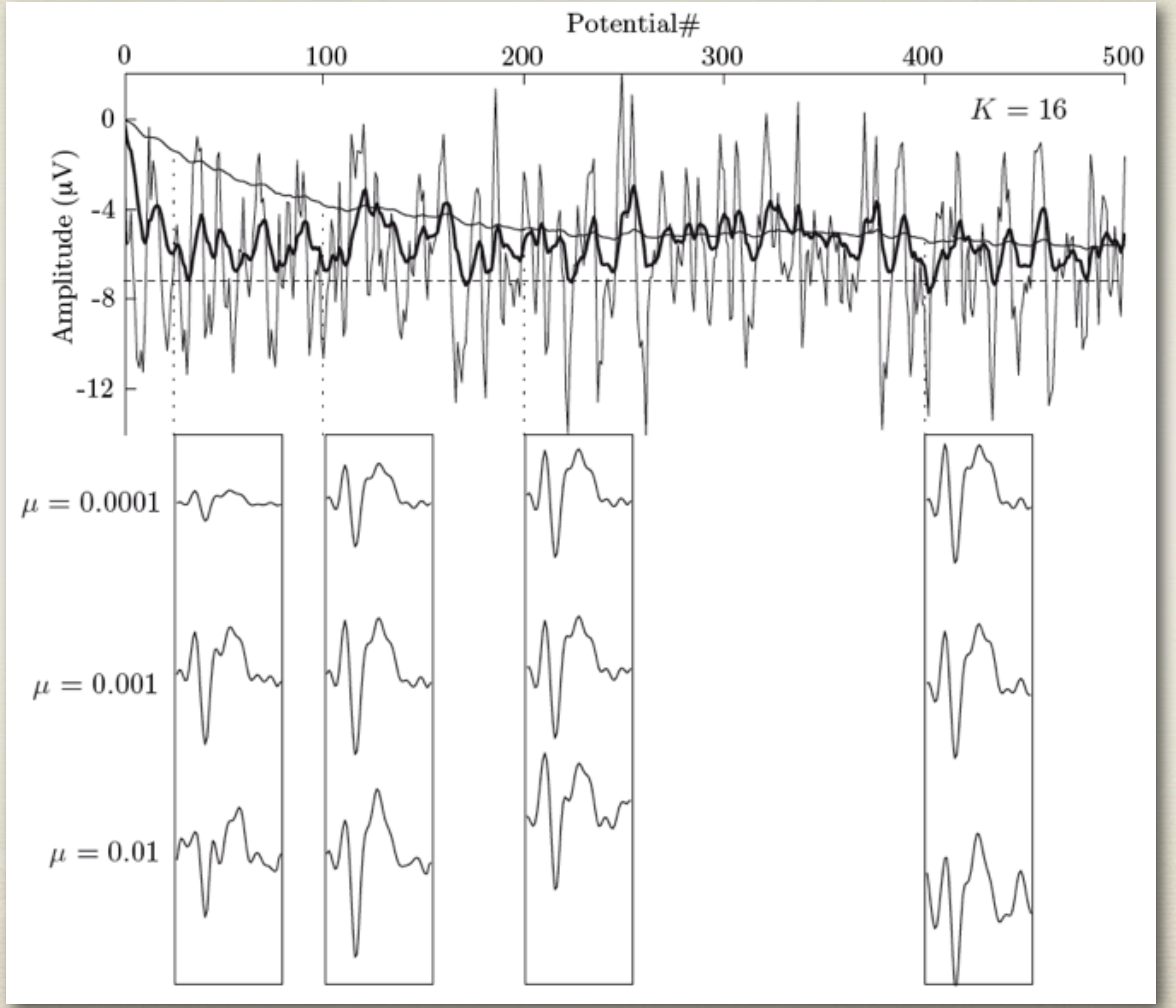


Figure 4.35: Adaptive linear combination of basis functions for the estimation of the weight vector $\mathbf{w}(n)$, using the observed signal $x(n)$.

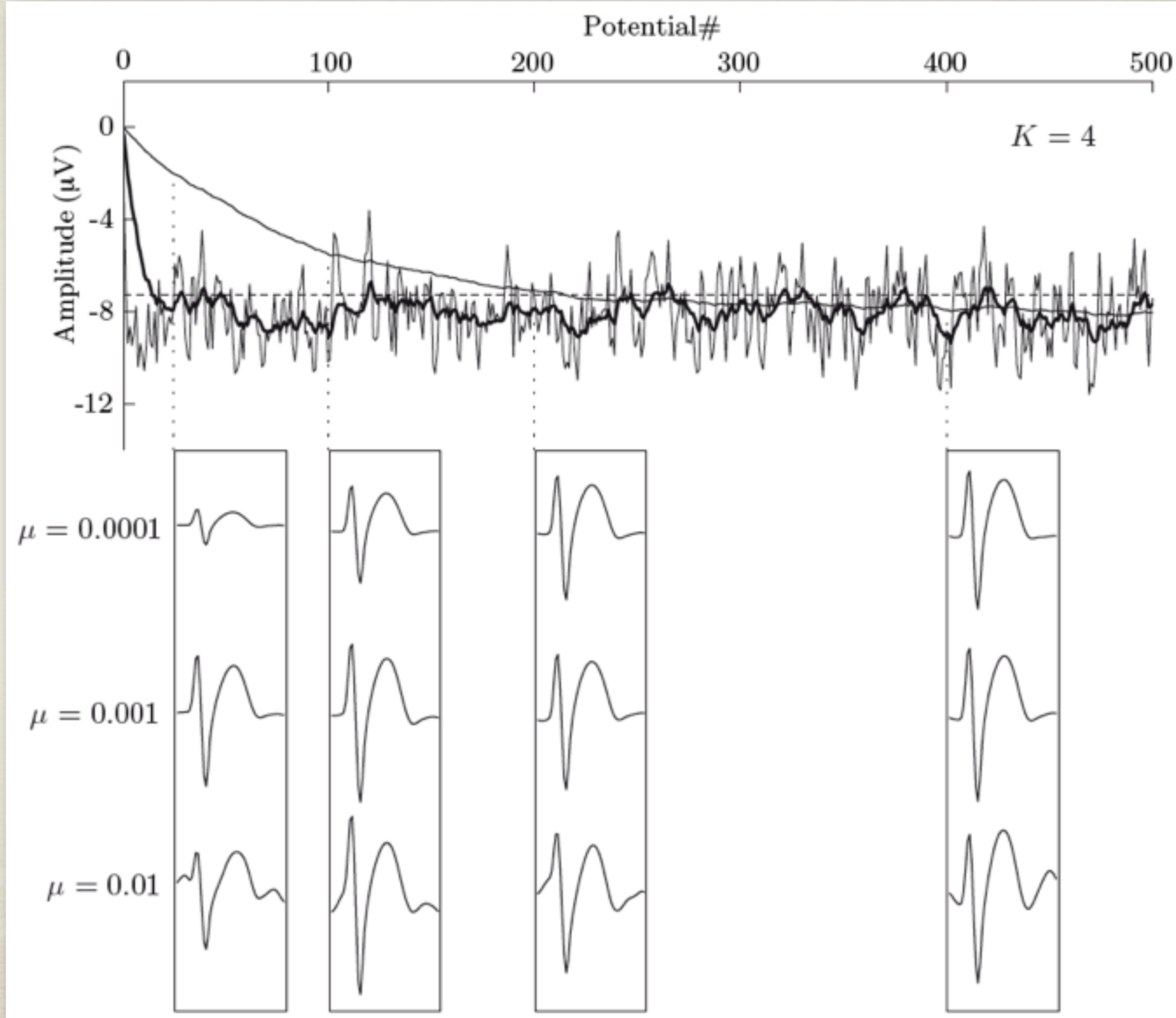
Adaptive Estimation of Weights

* The instantaneous LMS algorithm, in which the weights of the series expansion are adapted at every time instant, thereby producing a weight vector w(n)
* The block LMS algorithm, in which the weights are adapted only once for each EP ("block"), thereby producing a weight vector w_i that corresponds to the i:th potential.

Estimation Using Sine/Cosine



Estimation Using KL Functions



Limitations

* Sines/cosines and the KL basis functions lack the flexibility to efficiently track changes in latency of evoked potentials, i.e., changes in waveform width.
* The KL basis functions are not associated with any algorithm for fast computations since the functions are signal-dependent.

Wavelet Analysis

* Wavelets is a very general and powerful class of basis functions which involve two parameters: one for translation in time and another for scaling in time.
* The purpose is to characterize the signal with good localization in both time and frequency.
* These two operations makes it possible to analyze the joint presence of global waveforms ("large scale") as well as fine structures ("small scale") in a signal.
* Signals analyzed at different scales, with an increasing level of detail resolution, is referred to as a multi-resolution analysis.

Wavelet Applications

* signal characterization
* signal denoising
* data compression
* detecting transient waveforms
* and much more!

The Correlation Operation

Recall the fundamental operation in orthonormal basis function analysis: in discrete-time, the correlation between the observed signal x(n) and the basis functions $\varphi_k(n)$:

$$w_k = \sum_{n=0}^{N-1} x(n)\varphi_k(n),$$

In wavelet analysis, the two operations of scaling and translation in time are most simply introduced when the continuous-time description is adopted:

$$w_k = \int_{-\infty}^{\infty} x(t)\varphi_k(t)dt.$$

The Mother Wavelet

A family of wavelets $\psi_{s,\tau}(t)$ is defined by scaling and translating the *mother wavelet* $\psi(t)$ with the continuous-valued parameters s (> 0) and τ,

$$\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-\tau}{s}\right), \tag{4.298}$$

where … nergy.
Thus, … $s > 1$.

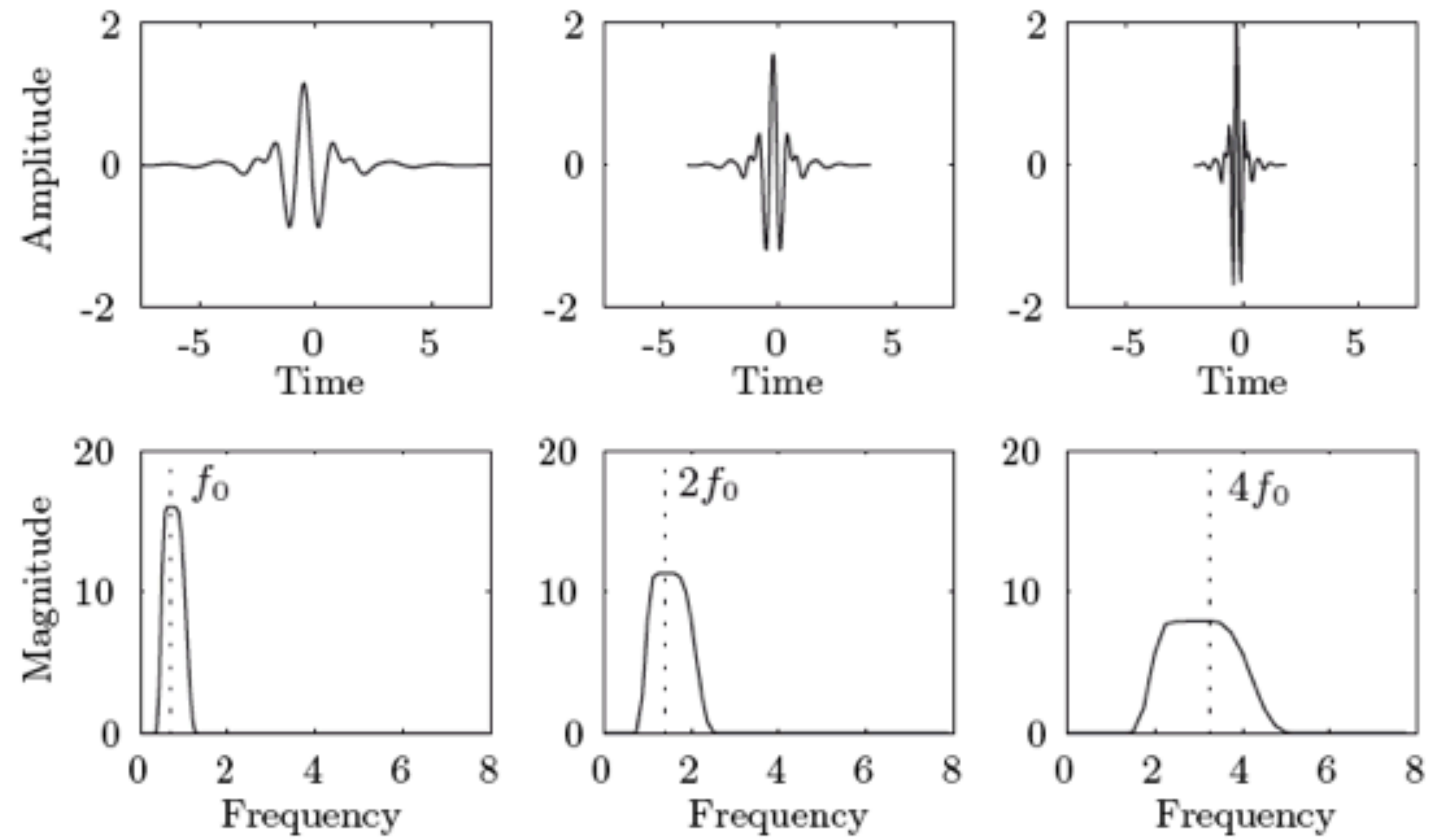


Figure 4.38: A wavelet shown at three different scales and the corresponding bandpass frequency responses (the Meyer wavelet). Note that the center frequency and bandwidth both increase as the wavelet is contracted in time.

The Wavelet Transform

C
W
T

The *continuous wavelet transform* (CWT) $w(s,\tau)$ of a continuous-time signal $x(t)$ is defined by the correlation between $x(t)$ and a scaled and translated version of $\psi(t)$,

$$w(s,\tau) = \int_{-\infty}^{\infty} x(t) \frac{1}{\sqrt{s}} \psi\left(\frac{t-\tau}{s}\right) dt, \tag{4.299}$$

I
C
W
T

The function $x(t)$ can be exactly recovered from $w(s,\tau)$ using the reconstruction equation [152]

$$x(t) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{0}^{\infty} w(s,\tau) \frac{1}{\sqrt{s}} \psi\left(\frac{t-\tau}{s}\right) \frac{d\tau ds}{s^2}, \tag{4.300}$$

where

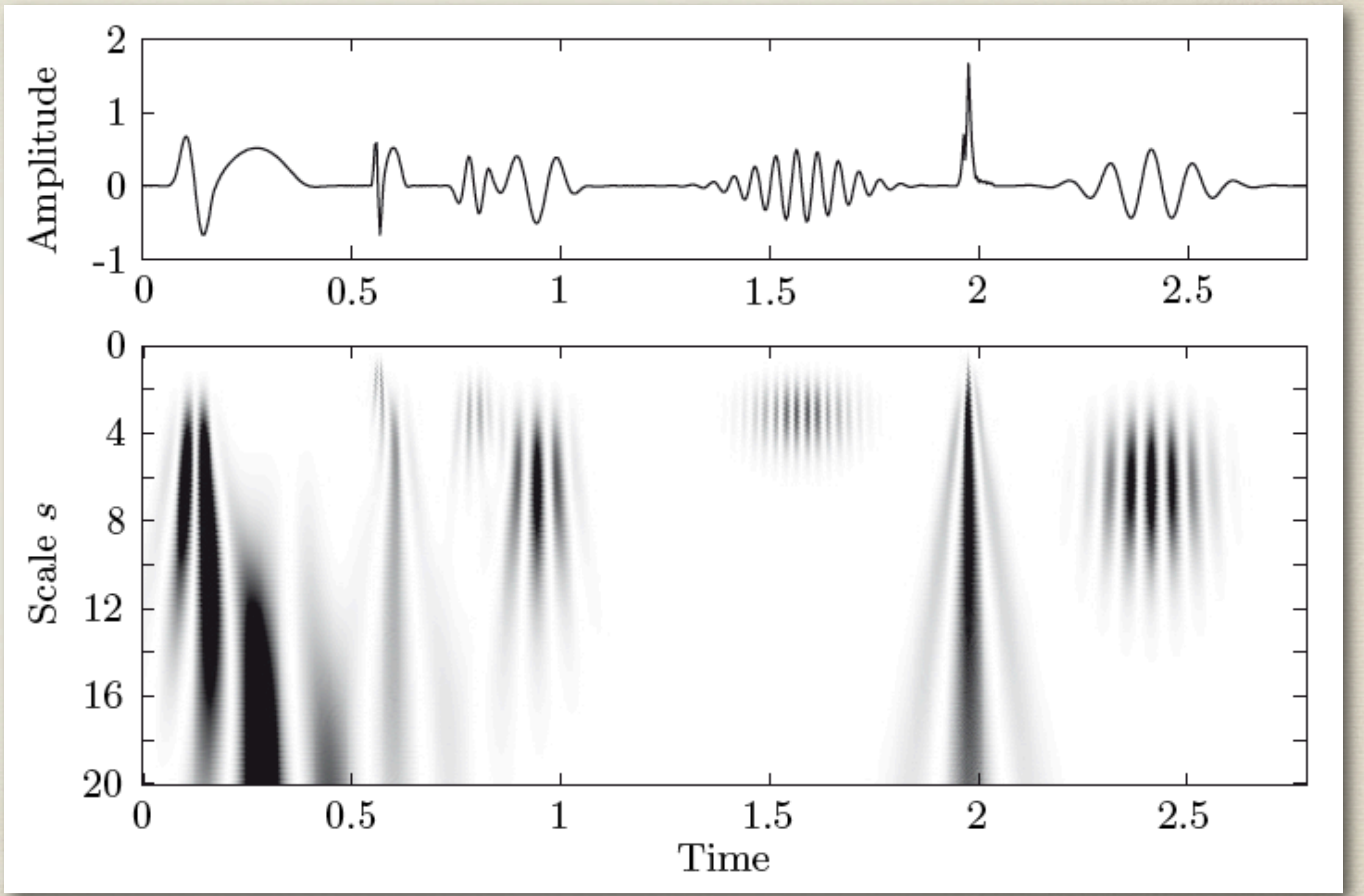
$$C_\psi = \int_{0}^{\infty} \frac{|\Psi(\Omega)|^2}{|\Omega|} d\Omega < \infty, \tag{4.301}$$

The Scalogram

Composite signal

Scalogram

$|w(s,\tau)|^2$



The Discrete Wavelet Transform

The CWT w(s, τ) is highly redundant
and needs to be sampled

Dyadic sampling

$$s = 2^{-j}, \quad \tau = k2^{-j},$$

The discretized wavelet function

$$\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k).$$

The discrete wavelet transform (DWT)

$$w_{j,k} = \int_{-\infty}^{\infty} x(t)\psi_{j,k}(t)dt.$$

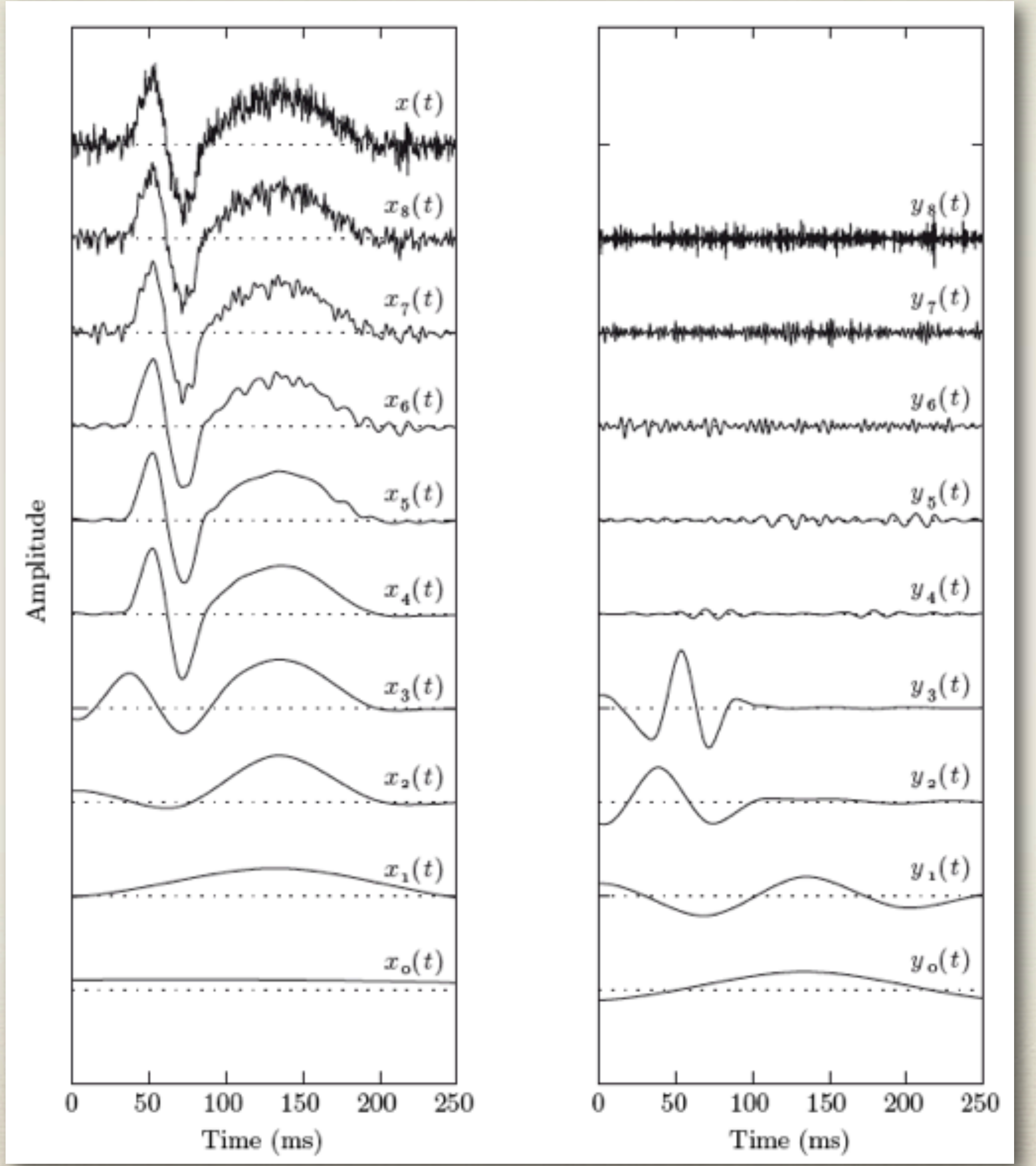
The inverse discrete wavelet transform (IDWT)

$$x(t) = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} w_{j,k}\psi_{j,k}(t),$$

Multiresolution Analysis

* A signal can be viewed as the sum of a smooth ("coarse") part and a detailed ("fine") part.
  * The smooth part reflects the main features of the signal, therefore called the approximation signal.
  * The faster fluctuations represent the signal details.
* The separation of a signal into two parts is determined by the resolution with which the signal is analyzed, i.e., by the scale below which no details can be discerned.

Multiresolution Analysis Exemplified



Multiresolution Analysis, cont'

In mathematical terms this is expressed as:

The approximation of a signal $x(t)$ at scale (resolution level) j is denoted $x_j(t)$. At the next scale $j+1$, the approximation signal $x_{j+1}(t)$ is composed of $x_j(t)$ and the details $y_j(t)$ at that level such that

$$x_{j+1}(t) = x_j(t) + y_j(t). \tag{4.309}$$

By adding more and more detail to $x_j(t)$ we arrive, as the resolution approaches infinity, at a dyadic multiresolution representation of the original signal $x(t)$ which involves a smooth part and the sum of different details,

$$x(t) = x_j(t) + \sum_{l=j}^{\infty} y_l(t). \tag{4.310}$$

The Scaling Function

* The scaling function $\varphi(t)$ is introduced for the purpose of efficiently representing the approximation signal $x_j(t)$ at different resolution.
* This function, being related to a unique wavelet function $\psi(t)$, can be used to generate a set of scaling functions defined by different translations:

$$\varphi_{0,k}(t) = \varphi(t - k),$$

where the index "0" indicates that no time scaling is performed.

The Scaling Function, cont'

The design of a scaling function φ(t) must be such that translations of φ(t) constitute an orthonormal set of functions, i.e.,

$$\int_{-\infty}^{\infty} \varphi_{0,k}(t)\varphi_{0,n}(t)dt = \int_{-\infty}^{\infty} \varphi(t-k)\varphi(t-n)dt \quad = \begin{cases} 1, & k = n; \\ 0, & k \neq n. \end{cases}$$

Its design is not considered in this course, but some existing scaling functions are applied.

The Approximation Signal $x_0(t)$

Therefore, the scaling functions $\varphi_{0,k}(t)$ are said to span a subspace $\mathcal{V}_0$ of the whole space of square integrable functions denoted $L^2(\mathbf{R})$,

$$\mathcal{V}_0 = \operatorname*{span}_k\{\varphi_{0,k}(t)\}. \tag{4.313}$$

The Approximation Signal $x_j(t)$

$$
\begin{aligned}
x_j(t) &= \sum_{n=-\infty}^{\infty} c_j(n)\varphi_{j,n}(t) \\
\text{(dyadic sampling)} &= 2^{j/2} \sum_{n=-\infty}^{\infty} c_j(n)\varphi(2^j t - n),
\end{aligned}
\tag{4.318}
$$

where

$$
c_j(k) = \int_{-\infty}^{\infty} x(t)\varphi_{j,k}(t)dt. \tag{4.319}
$$

It is important to realize that, for $j > 0$, the span increases since $\varphi_{j,k}(t)$ contracts in time, thereby allowing details of $x(t)$ to be better represented by the approximation signal $x_j(t)$. On the other hand, only the coarser information can be represented for $j < 0$ since $\varphi_{j,k}(t)$ then expands.

The Multiresolution Property

The subspace $\mathcal{V}_j$ is spanned by $\varphi_{j,k}(t)$,

$$\mathcal{V}_j = \operatorname*{span}_k \{\varphi_{j,k}(t)\}, \tag{4.320}$$

which has a time resolution only half as good as that of $\mathcal{V}_{j+1}$ since the scaling function in $\mathcal{V}_{j+1}$ is contracted by a factor of two, i.e., $\varphi(2^{j+1}t)$ in relation to $\varphi(2^j t)$.

Each subspace is spanned by a different set of basis functions $\varphi_{j,k}(t)$, offering progressively better approximations such that $x_j(t)$ approaches $x(t)$ in the limit as $j \to \infty$,

$$\lim_{j\to\infty} x_j(t) = x(t), \tag{4.322}$$

The Refinement Equation

$$\begin{aligned}\varphi(t) &= \sum_{n=-\infty}^{\infty} h_\varphi(n)\varphi_{1,n}(t)\\ &= \sqrt{2}\sum_{n=-\infty}^{\infty} h_\varphi(n)\varphi(2t-n),\end{aligned}$$

$h_\varphi(n)$ is a sequence of scaling coefficients

The Wavelet Function

* It is desirable to introduce the function ψ(t) which complements the scaling function by accounting for the details of a signal rather than its approximations.
* For this purpose, a set of orthonormal basis functions at scale j is given by

$$\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k),$$

which spans the difference between the two subspaces Vj and Vj+1.

Scaling and Wavelet Functions

At scale $j+1$, the subspace describing signal detail is given by

$$\mathcal{W}_j = \operatorname*{span}_k \{\psi_{j,k}(t)\}, \tag{4.325}$$

where the wavelet functions that span $\mathcal{W}_j$ are required to be orthonormal to the scaling functions of $\mathcal{V}_j$,

$$\int_{-\infty}^{\infty} \varphi_{j,k}(t)\psi_{j,l}(t)dt = 0, \tag{4.326}$$

for all indices j and k.

Orthogonal Complements

In the subspace $\mathcal{V}_{j+1}$, $\mathcal{W}_j$ is said to constitute an orthogonal complement to $\mathcal{V}_j$ which is denoted

$$\mathcal{V}_{j+1} = \mathcal{V}_j \oplus \mathcal{W}_j, \tag{4.327}$$

where $\oplus$ denotes the direct sum between two subspaces. Since (4.327) is valid for an arbitrary value of j, we also have that

$$\mathcal{V}_j = \mathcal{V}_{j-1} \oplus \mathcal{W}_{j-1}, \tag{4.328}$$

which, when continued until a certain value j_0 ($\leq j$) is reached, yields the decomposition

$$\mathcal{V}_{j+1} = \mathcal{V}_{j_0} \oplus \mathcal{W}_{j_0} \oplus \mathcal{W}_{j_0+1} \oplus \ldots \oplus \mathcal{W}_j. \tag{4.329}$$

As j approaches infinity, the subspace decomposition can be expressed as

$$x(t) = x_{j_0}(t) + \sum_{j=j_0}^{\infty} y_j(t), \tag{4.330}$$

The Wavelet Series Expansion

a *wavelet series expansion* in terms of the scaling coefficients $c_{j_0}(k)$ and the wavelet coefficients $d_j(k)$,

$$x(t) = \sum_{n=-\infty}^{\infty} c_{j_0}(n)\varphi_{j_0,n}(t) + \sum_{j=j_0}^{\infty}\sum_{n=-\infty}^{\infty} d_j(n)\psi_{j,n}(t). \tag{4.333}$$

Compare this expansion with the orthogonal expansions mentioned earlier such as the one with sine/cosine basis functions, i.e., the Fourier series. The wavelet/scaling coefficients do not have a similar simple interpretation.

Multiresolution Signal Analysis: A Classical Example

The Haar
scaling function

$$\varphi(t) = \begin{cases} 1, & 0 \le t < 1; \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{bmatrix} h_\varphi(0) & h_\varphi(1) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

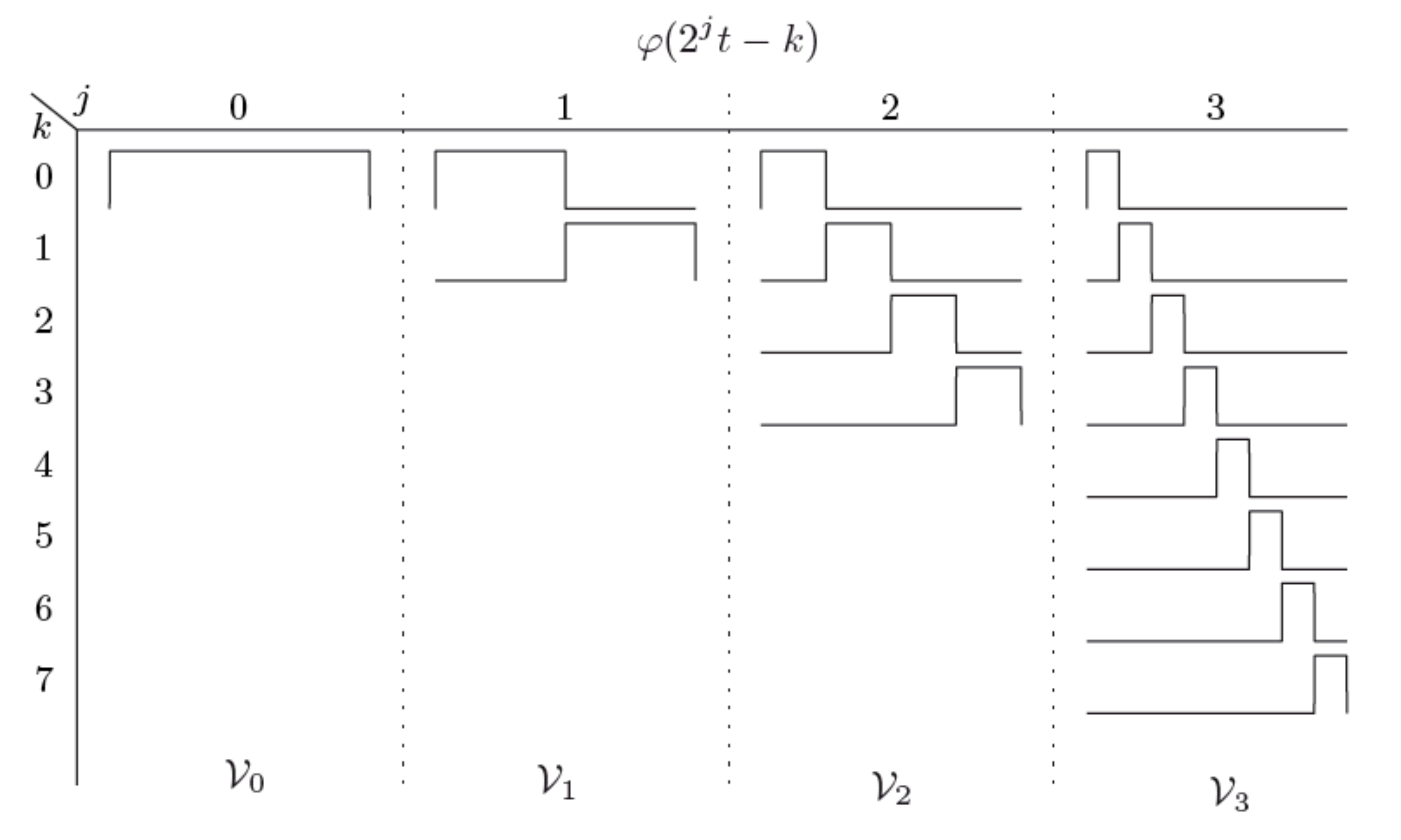
The Haar
wavelet function

$$\psi(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2}; \\ -1, & \frac{1}{2} \le t < 1; \\ 0, & \text{otherwise,} \end{cases}$$

$$\begin{bmatrix} h_\psi(0) & h_\psi(1) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

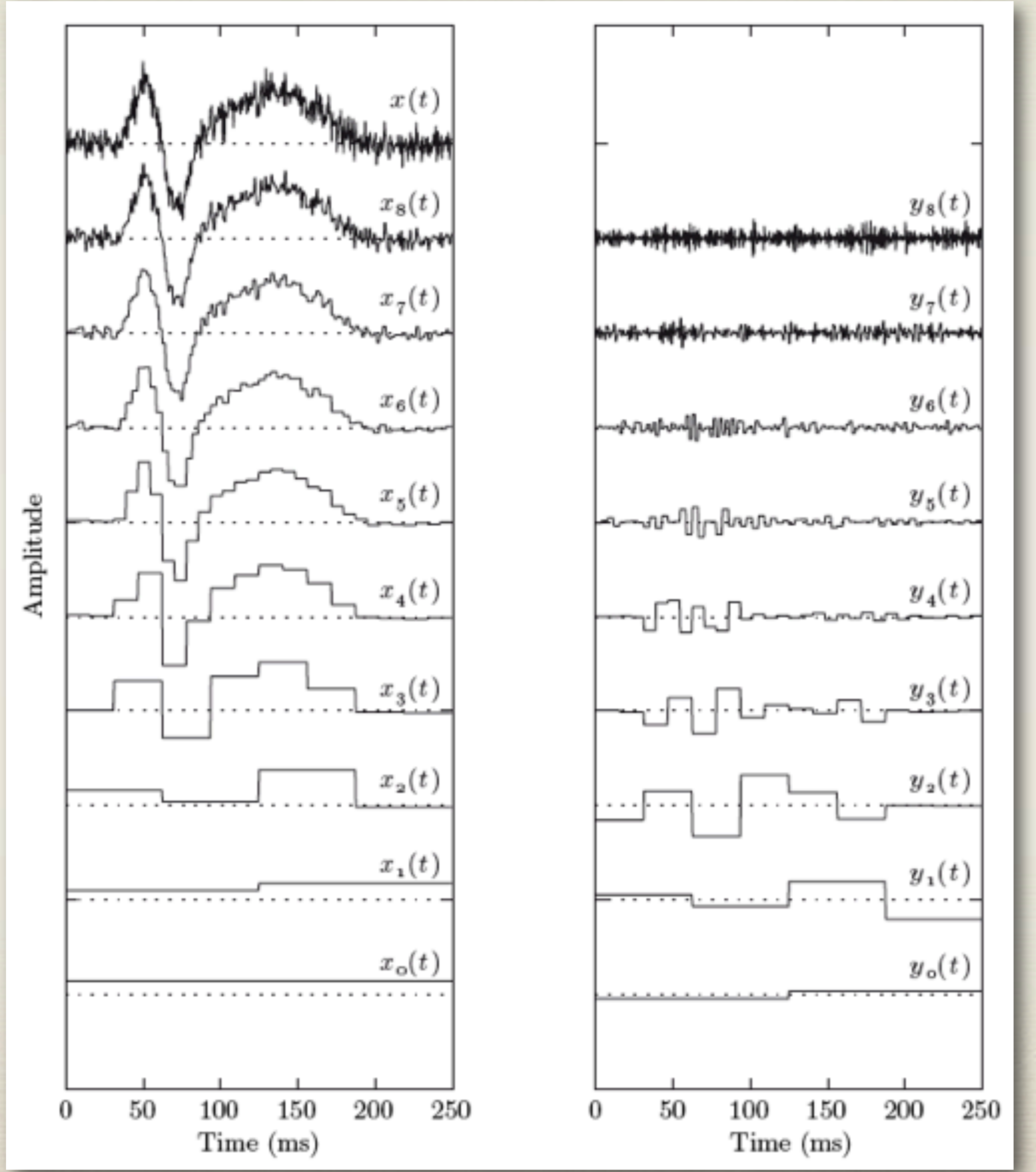
These functions are individually and mutually orthonormal

The Haar Scaling Function



Haar Multiresolution Analysis

Approximation signals



Detail signals

Haar Scaling and Wavelet Functions

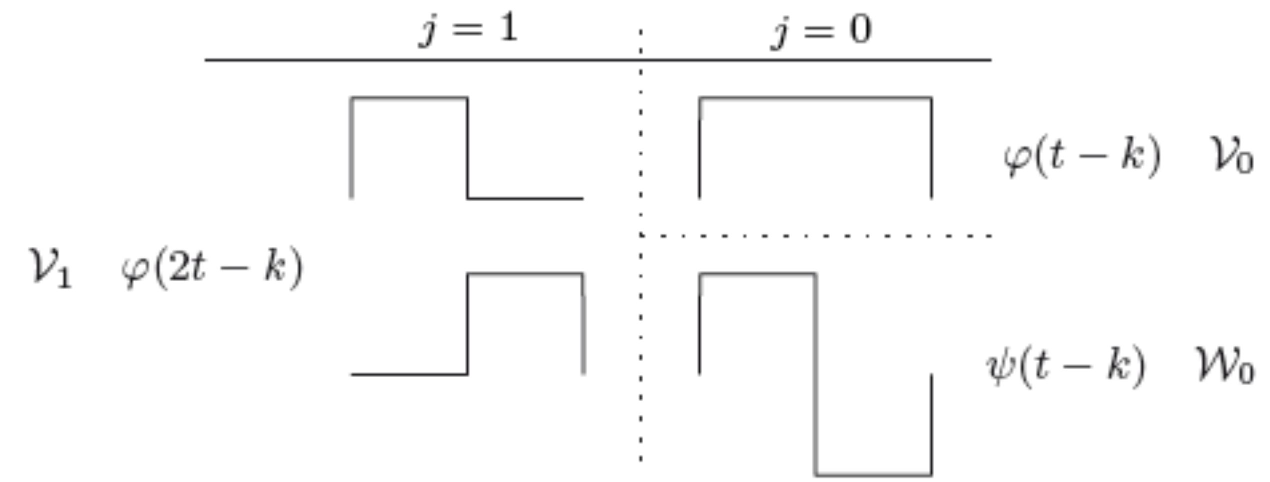


Figure 4.42: Decomposition of the subspace $\mathcal{V}_1 = \mathcal{V}_0 \oplus \mathcal{W}_0$ by the Haar scaling and wavelet functions.

Computation of Coefficients

The scaling and wavelet coefficients can computed recursively by exploring the refinement equation

$$\varphi(t) = \sum_{n=-\infty}^{\infty} h_\varphi(n)\varphi_{1,n}(t)$$
$$= \sqrt{2}\sum_{n=-\infty}^{\infty} h_\varphi(n)\varphi(2t-n),$$

so that, for example, the scaling coefficients are computed with

$$c_j(k) = \sum_{n=-\infty}^{\infty} h_\varphi(n-2k)c_{j+1}(n)$$
$$= h_\varphi(-n) * c_{j+1}(n)|_{n=2k}.$$

see derivation on page 300

Filter Bank Implementation

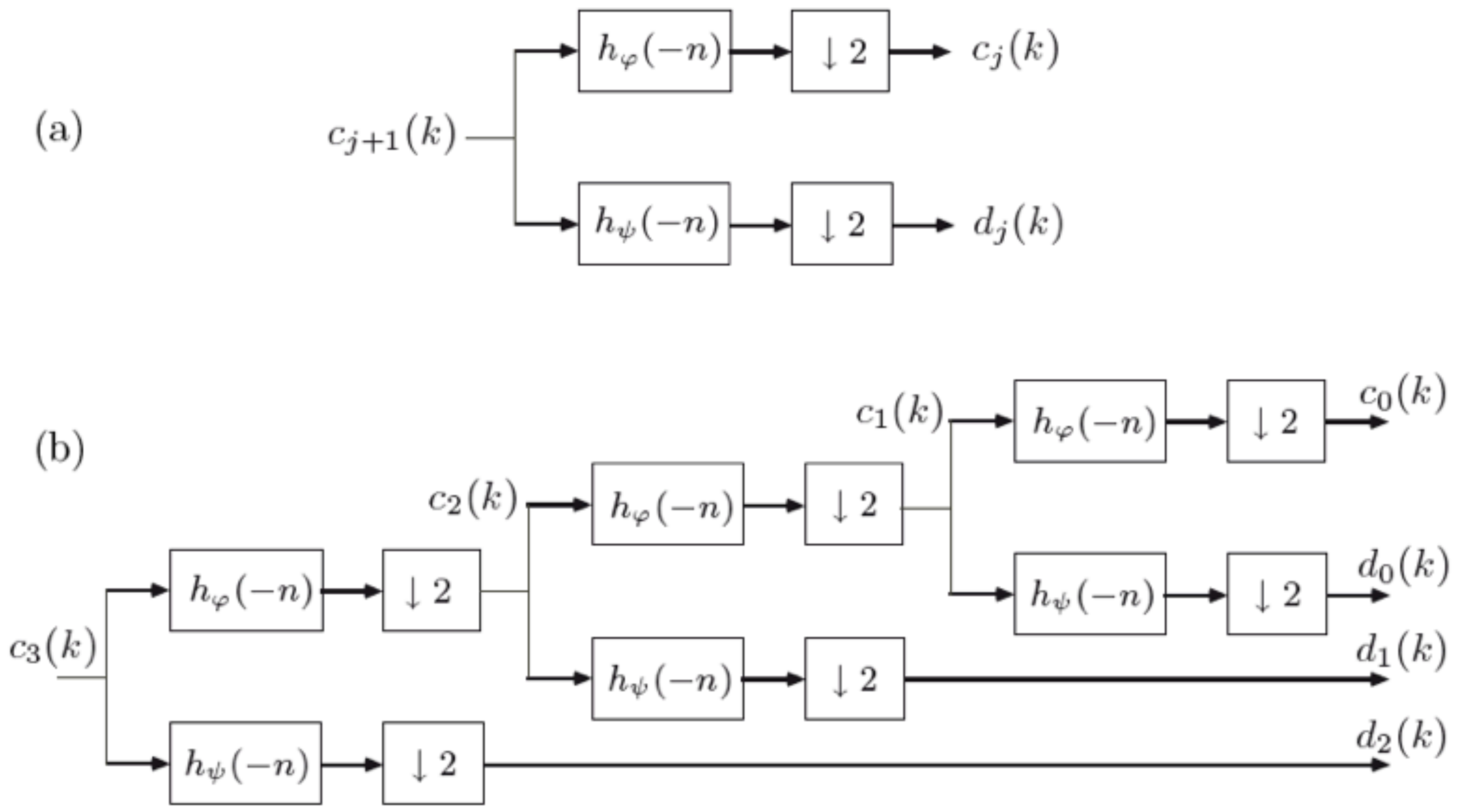


Figure 4.43: (a) A two-channel analysis filter bank for calculating the coefficients of the wavelet series expansion in (4.333). (b) The discrete wavelet transform based on the filter bank in (a), which, in this case, produces the coefficients that decompose the space $\mathcal{V}_3$ into $\mathcal{V}_0, \mathcal{W}_0, \mathcal{W}_1$, and $\mathcal{W}_2$.

DWT Calculation

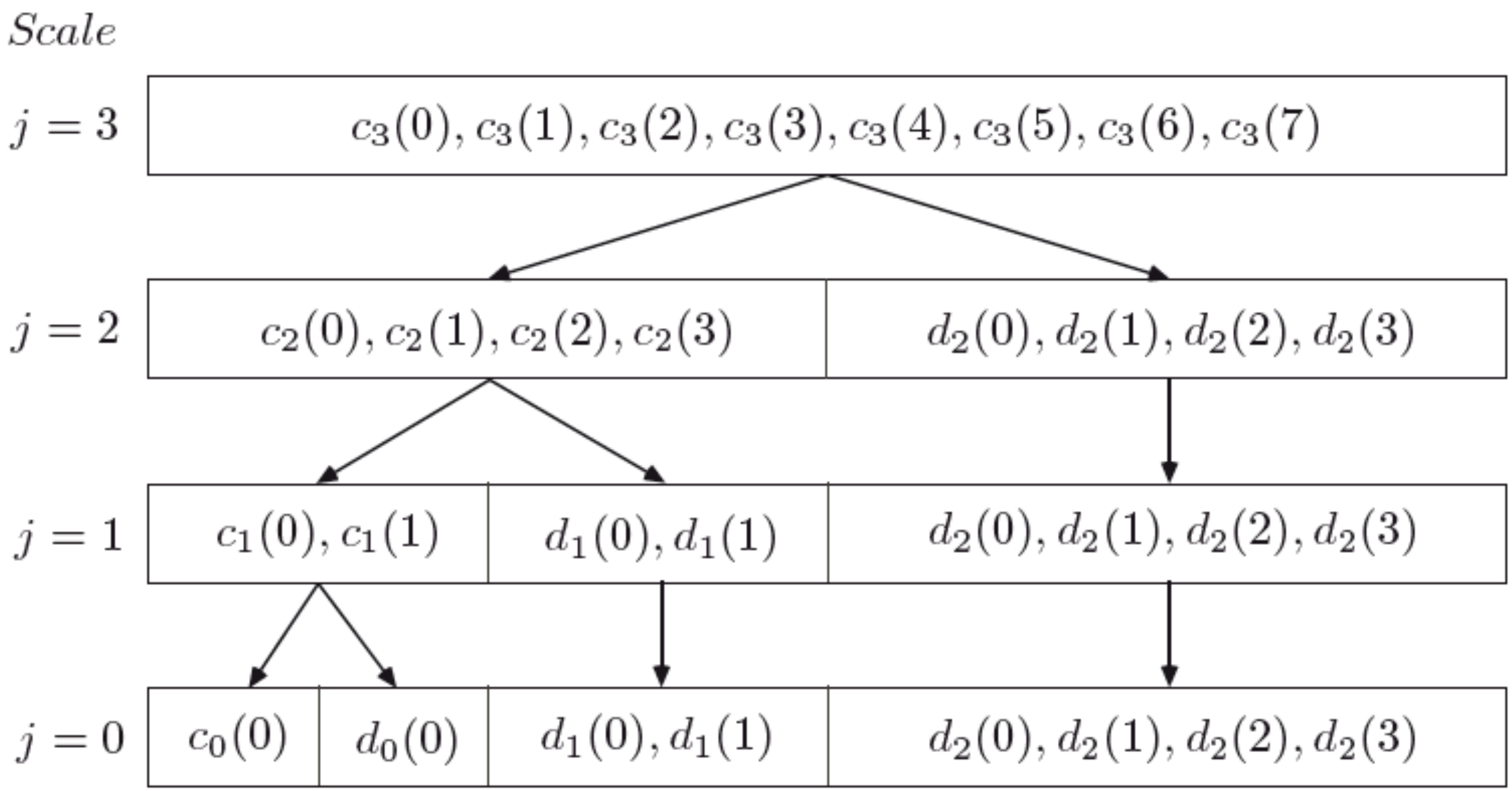


Figure 4.44: Calculation of the DWT for a signal of length $N = 8$. The final result is given by the coefficients at the bottom for $j = 0$. The vertical arrows indicate that the coefficients are simply copied down from the previous scale. The calculation is initialized by setting the coefficients $c_3(k)$ equal to the signal samples $x(k)$.

Inverse DWT Calculation

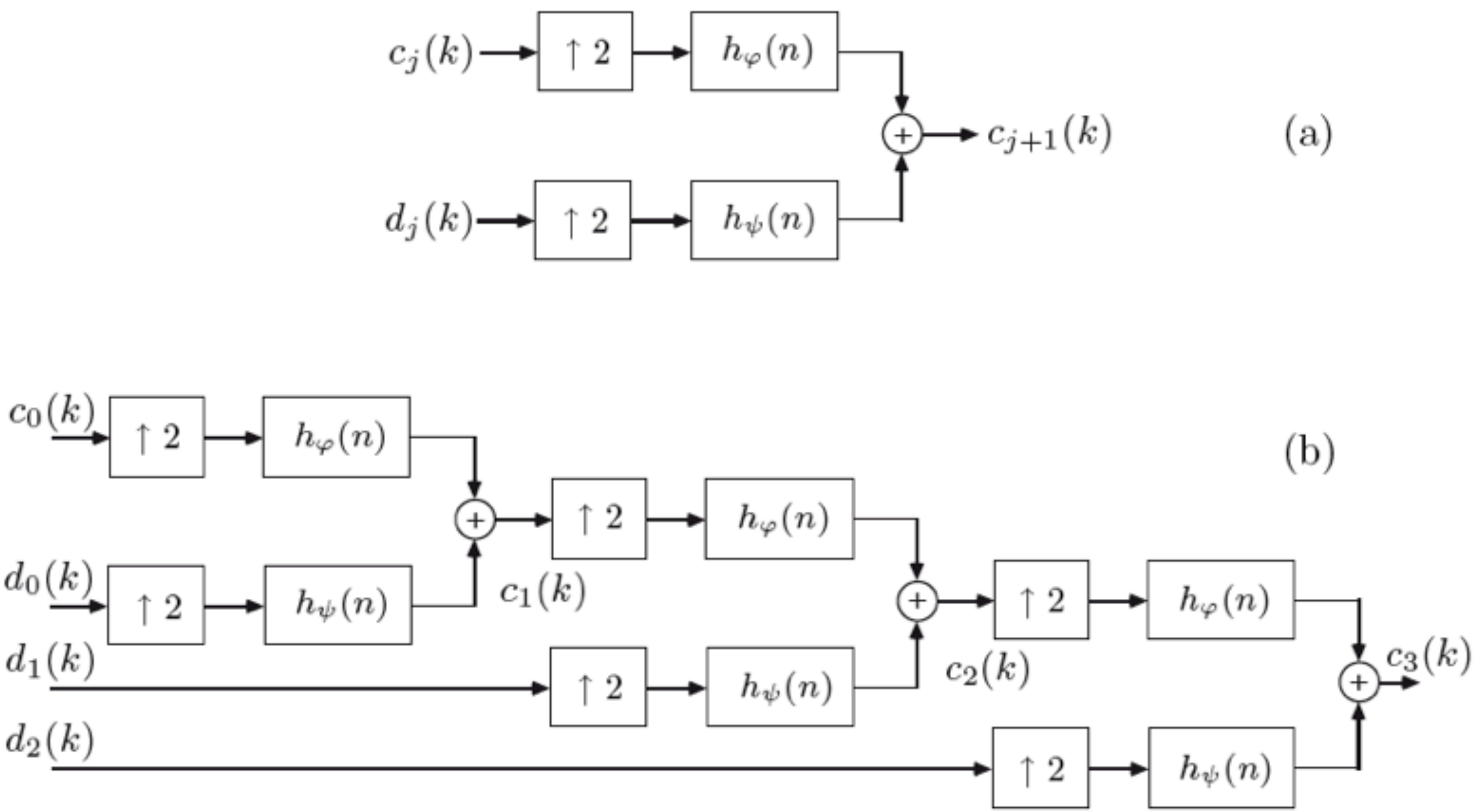


Figure 4.45: (a) A two-channel synthesis filter bank. (b) The inverse discrete wavelet transform based on the filter bank in (a) which, in this case, produces the coefficients of the space $\mathcal{V}_3$ based on $\mathcal{V}_0, \mathcal{W}_0, \mathcal{W}_1$, and $\mathcal{W}_2$.

Scaling Function Examples

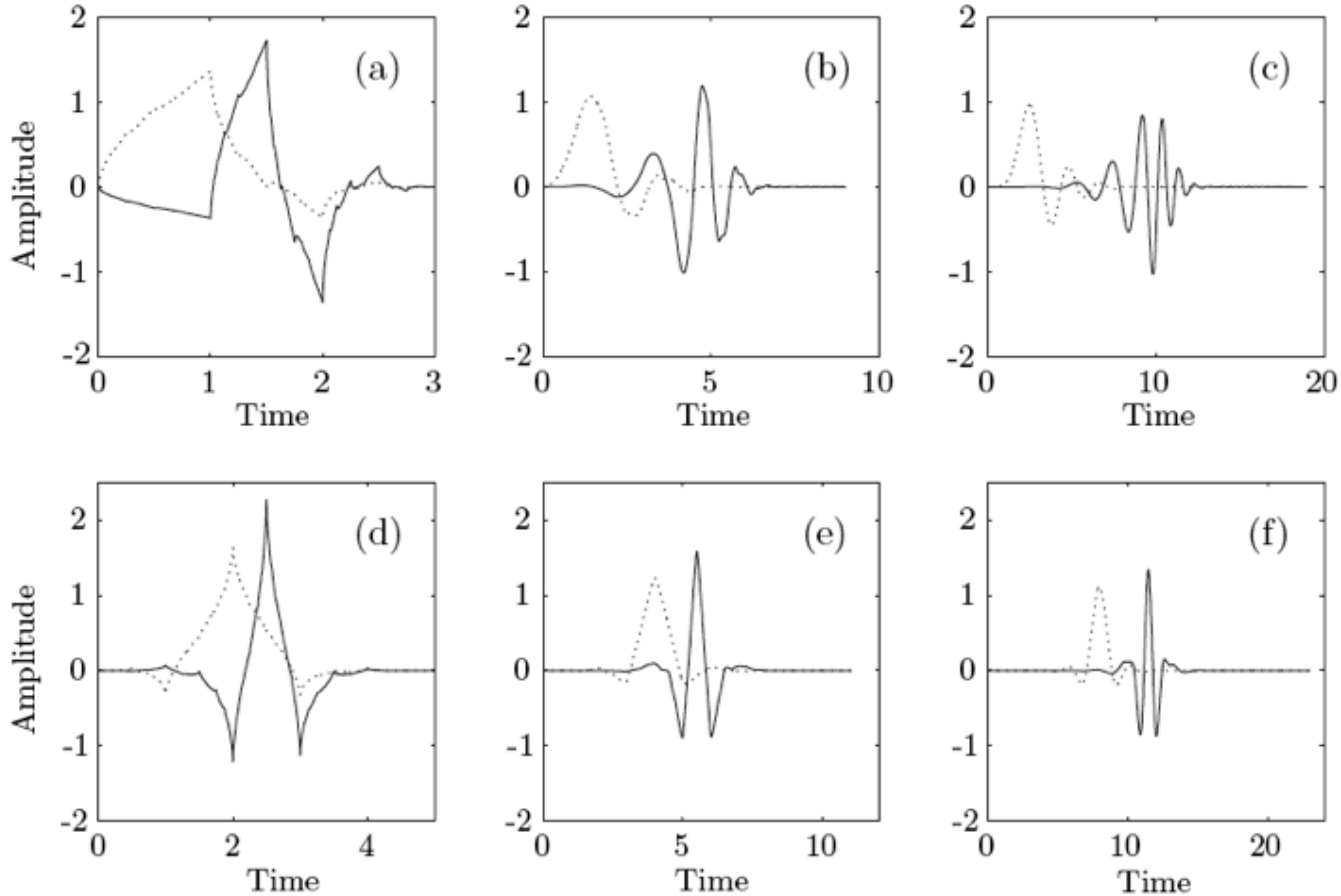
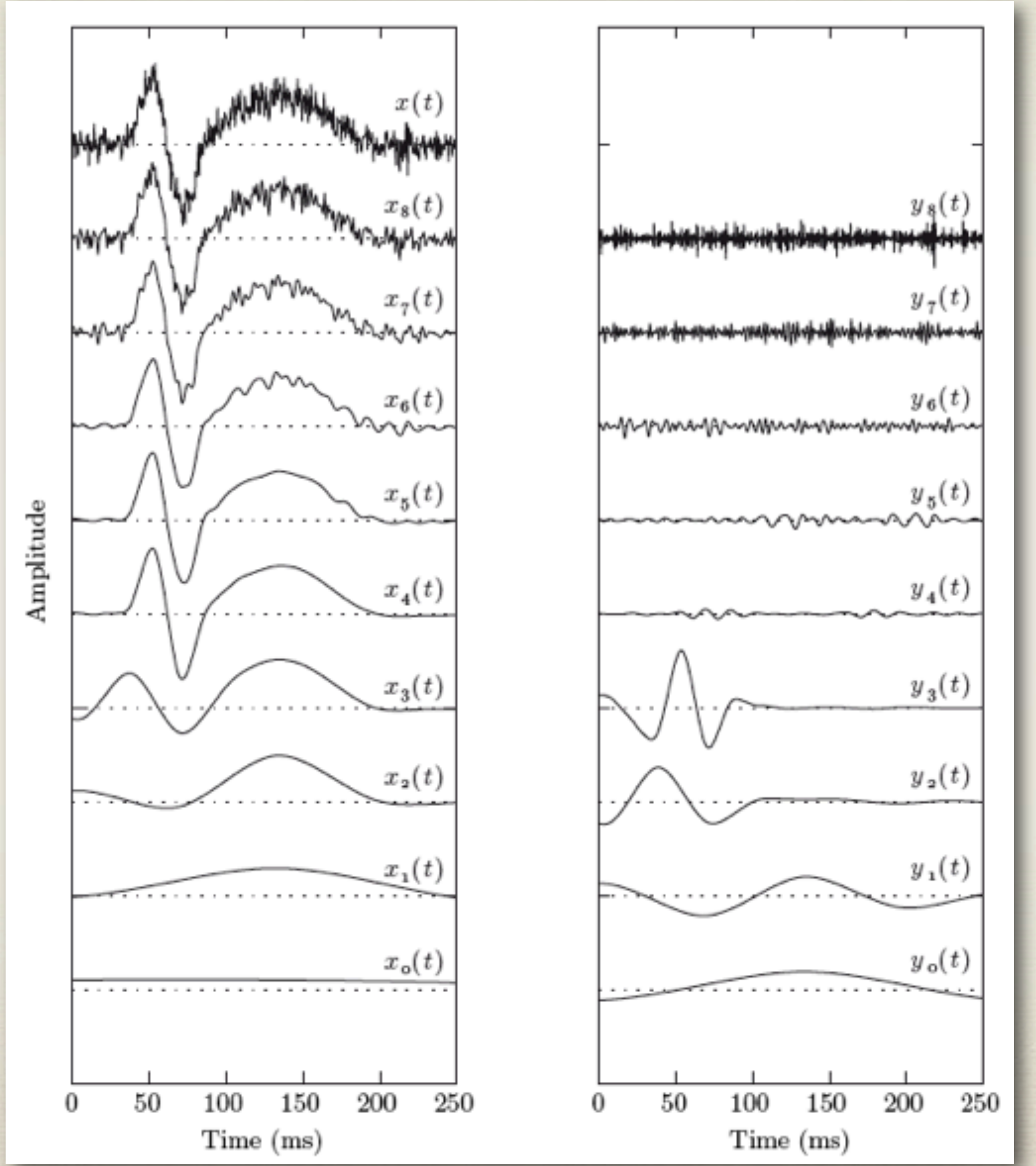
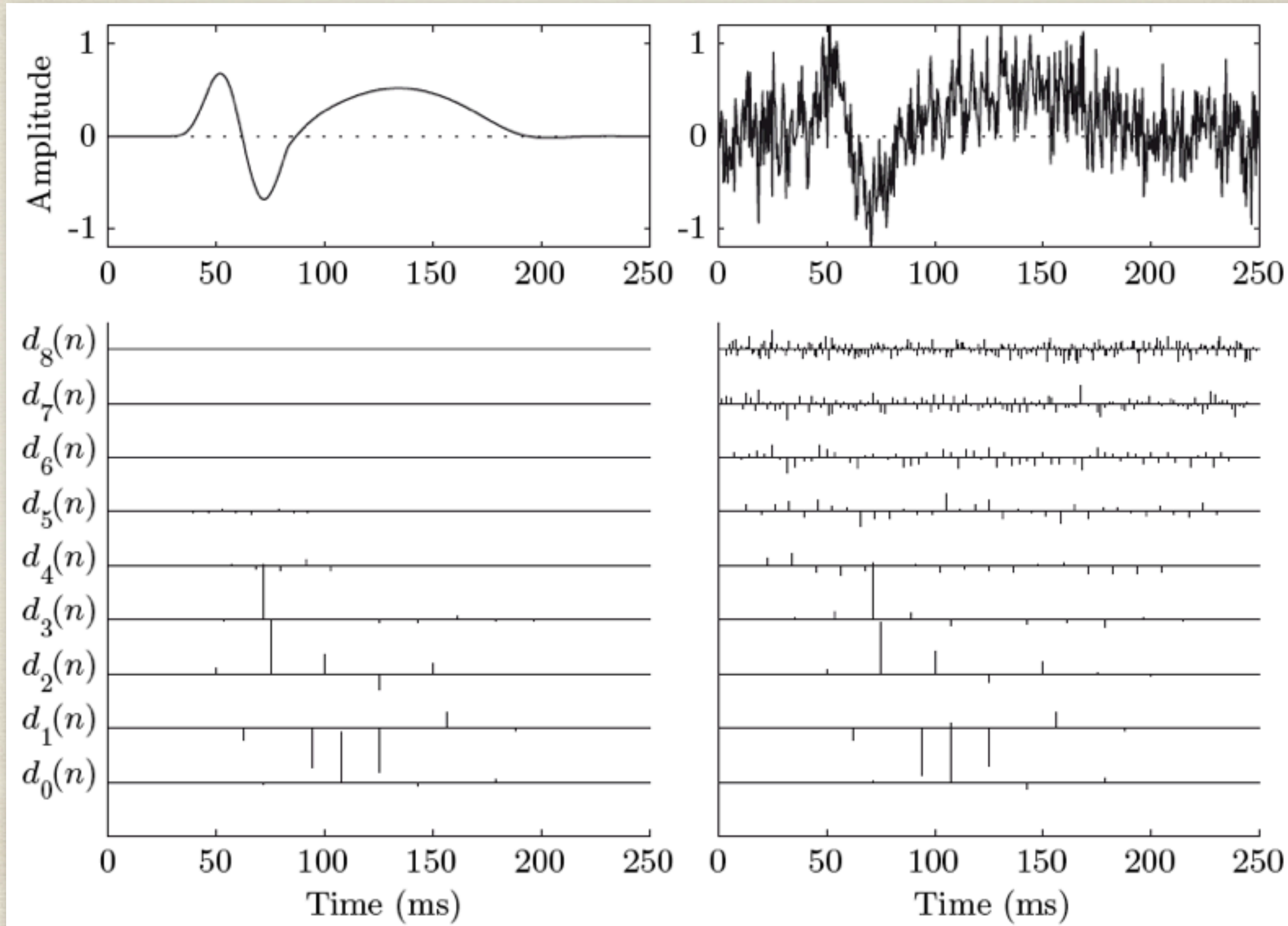


Figure 4.46: The scaling function (dotted line) and wavelet function (solid line) for (a) Daubechies–2, (b) Daubechies–5, (c) Daubechies–10, (d) Coiflet–1, (e) Coiflet–2, and (f) Coiflet–4. Note that the timescale differs between the diagrams.

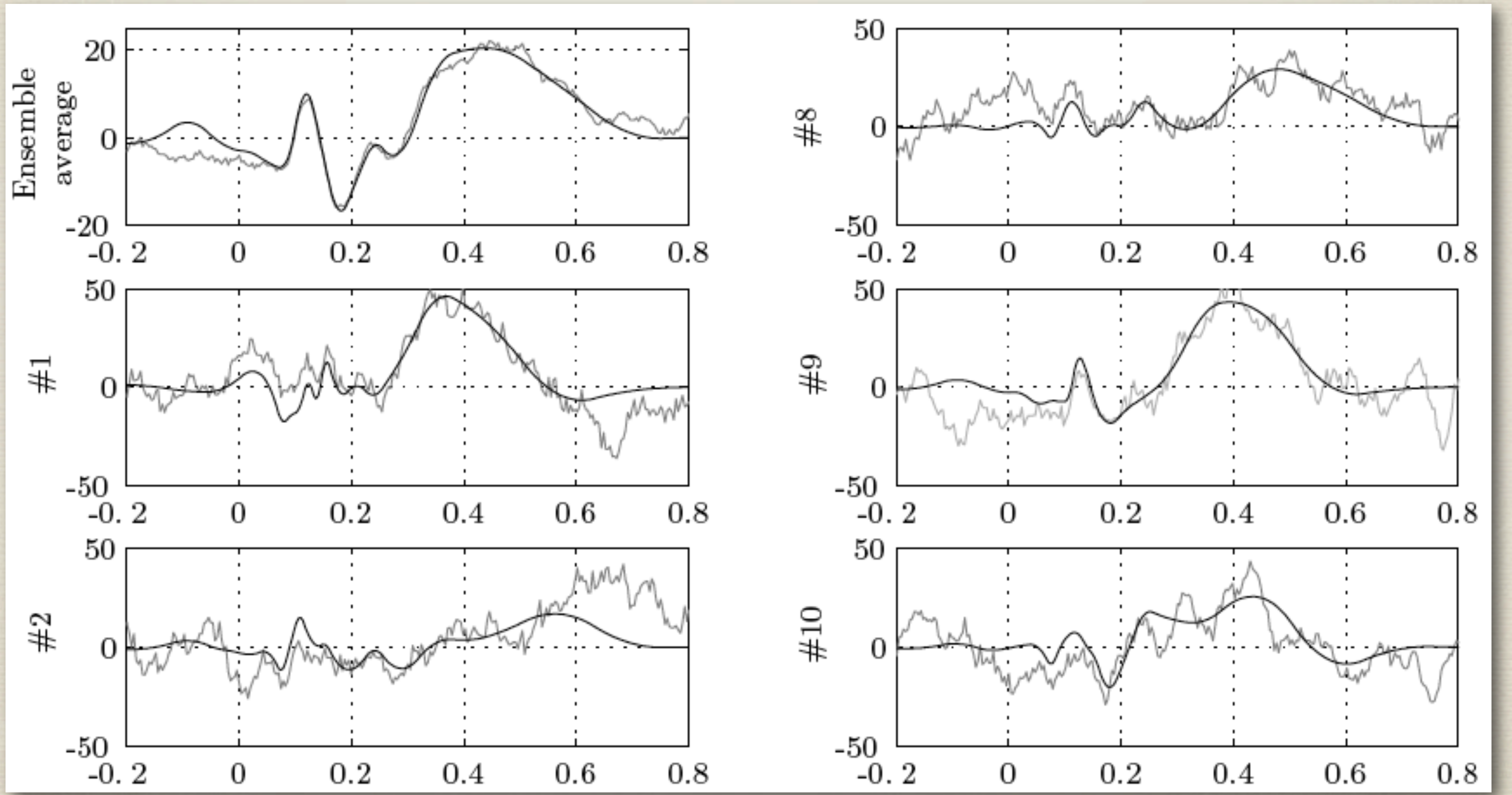
Coiflet Multiresolution Analysis



Scaling Coefficients in Noise



Denoising of Evoked Potentials



EP Wavelet Analysis

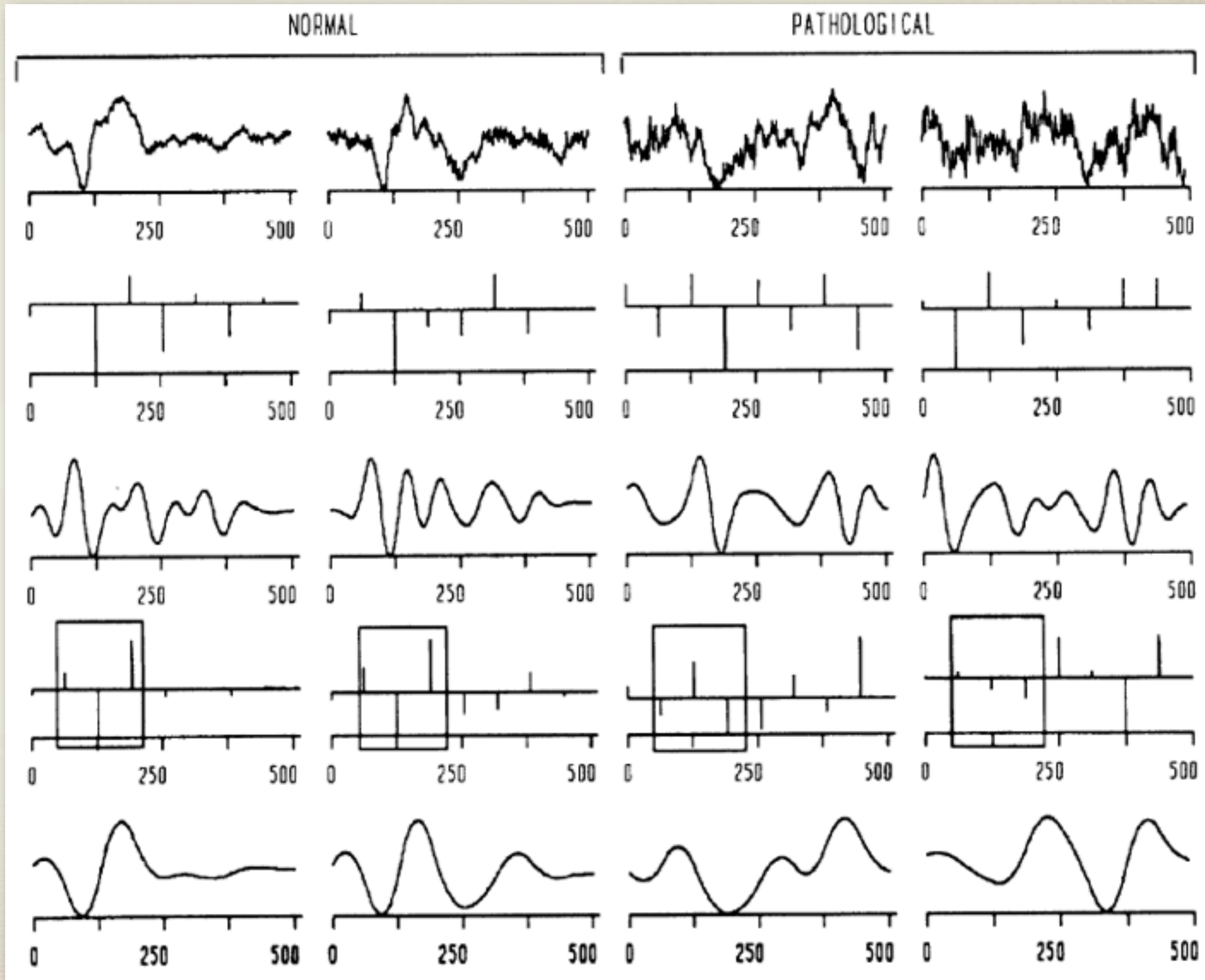
Visual EP

coefficients of W_3

reconstructed waveform

coefficients of V_3

reconstructed waveform



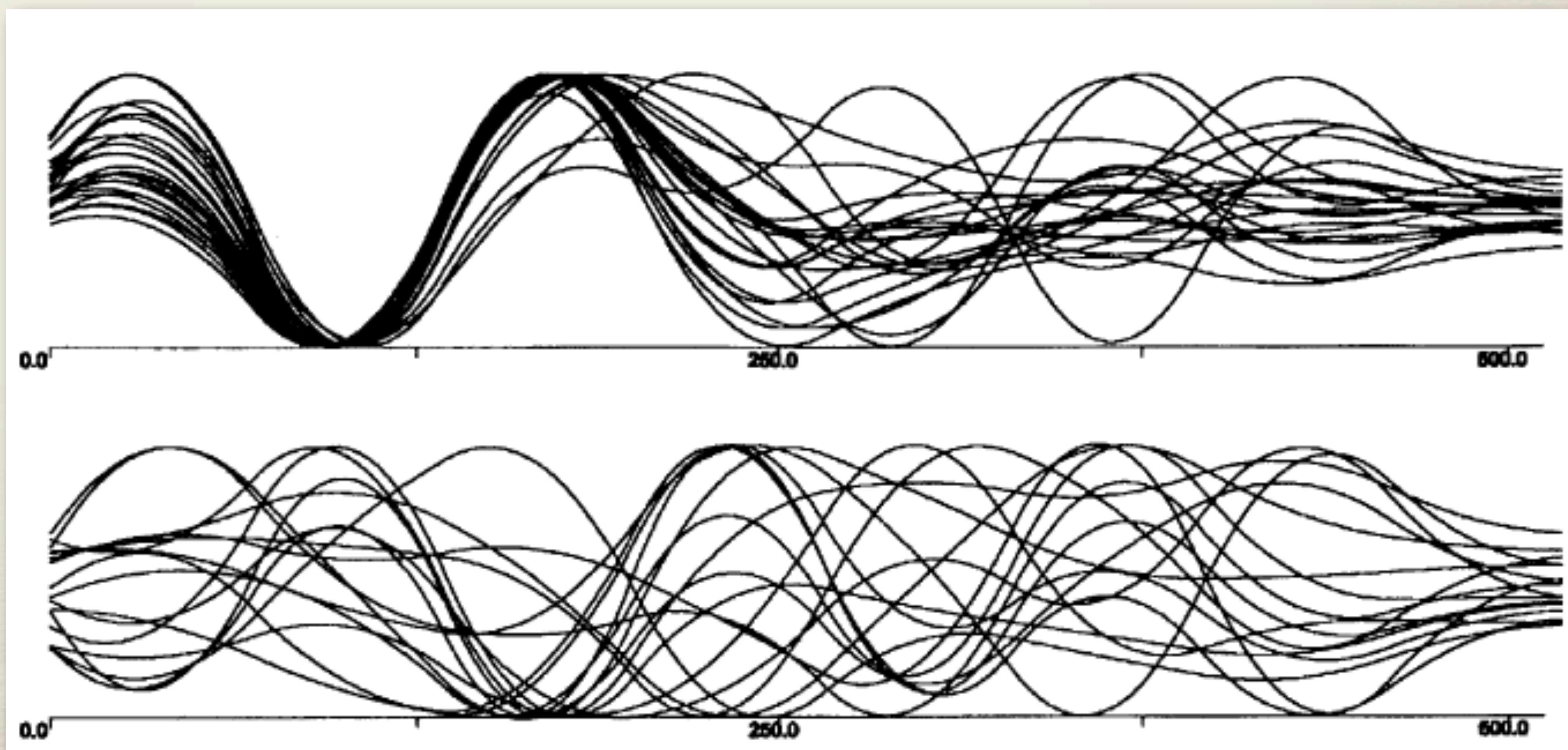
from Ademoglu et al., 1997

EP Wavelet Analysis, cont'

Waveforms reconstructed from V_3 and superimposed for 24 normal subjects (upper panel) and for 16 patients with dementia (lower panel).

Normal

Dement



EMG

IS NOT COVERED IN THIS COURSE